Curiosando con la matematica
Browsing through Mathematics

REVERSE MATHEMATICS

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Reverse mathematics is a branch of mathematical logic that was first introduced by Harvey Friedman in the 70s.

In the 90s an increasing number of mathematicians became familiar with it.

Nowadays reverse mathematics is a well-known field of mathematical logic. Many techniques are used in this field and they come from various branches of mathematical logic.

Definition

A first order theory T is a set of sentences in a first-order language $\mathcal{L}$. A sentence is a formula of $\mathcal{L}$ without free variables.

Examples:

- $\mathcal{L} = \{0, 1, +, <\}$, $\forall n(n + 1 \neq 0)$.
- $n < m$ is not a statement.

PEANO ARITHMETIC

$\mathcal{L} = \{0, +, \times, S, =\}$. The axioms are:

1. $\forall n \neg(S(n) = 0)$.
2. $\forall n \forall m (S(n) = S(m) \Rightarrow n = m)$.
3. $\forall n (n + 0 = n)$.
4. $\forall n \forall m (n + S(m) = S(n + m))$.
5. $\forall n (n \times 0 = 0)$.
6. $\forall n \forall m (n \times S(m) = ((n \times m) + n))$.
7. The scheme of induction:
   $\varphi(0) \wedge [\forall n (\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)]$.

But this theory is too weak to express most of mathematical statements!

$\mathcal{L} = \{=, <, \in, +, \cdot, 0, S\}$. We have a two-tiered system of variables $x, y, z, \ldots$ that range over natural numbers and $X, Y, Z, \ldots$ that range over sets of natural numbers. The axioms are:

1. $\forall n(n + 1 \neq 0)$.
2. $\forall n \forall m(S(n) = S(m) \Rightarrow n = m)$.
3. $\forall n(n + 0 = n)$.
4. $\forall n \forall m(n + S(m) = S(n + m))$.
5. $\forall n(n \times 0 = 0)$.
6. $\forall n \forall m(n \times S(m) = ((n \times m) + n))$.
7. $\forall m \neg(m < 0)$.
8. $\forall m \forall n(m < S(n) \Leftrightarrow (m < n \vee m = n)$.

These ones are called *arithmetic axioms*.

Scheme of comprehension:

$$\exists X \forall n (n \in X \Leftrightarrow \varphi(n)).$$

Scheme of induction:

$$(\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(S(n))) \Rightarrow \forall n \varphi(n)).$$

Z_2 stands for the second order arithemtic.

Weyl, Hilbert and Bernays were among the first to show that in Z_2 it is possible to formalize a considerable part of ordinary mathematics.

However, how is it possible to speak about real numbers, continuous functions, partial orders, ordinals, and so on, in a language in which we only have elements and subsets of $\mathbb{N}$?

We use codes, as we did in ZF. For example, to define $\mathbb{R}$, that is uncountable, we observe that $\mathbb{R}$ is the completion of $\mathbb{Q}$, which is countable.

MAIN QUESTION:

Given a theorem τ of ordinary mathematics, what is the weakest subsystem of Z_2 in which τ is provable?

Which axioms of Z_2 are needed to prove τ?

Five answers (BIG FIVE): $RCA_0, WKL_0, ACA_0, ATR_0, \Pi^1_1\text{-}CA_0$.

The full comprehension schema is replaced by comprehension only for Δ^0_1 sets.
The induction schema is limited to Σ^0_1 formulae.

$RCA_0 \vdash$:

1. Baire category theorem.
2. Intermediate value theorem.
3. Soundness theorem.
4. Every field has an algebraic closure.

RCA_0 is the base system for reverse mathematics.

$\mathsf{WKL}_0 = \mathsf{RCA}_0 +$ "every infinite tree of $2^{<\mathbb{N}}$ has a path".

$\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \Leftrightarrow \tau$, where τ could be:

1. Heine-Borel theorem: every covering of $[0,1]$ by a sequence of open intervals has a finite subcovering.
2. Every continuous function $f : [0,1] \to \mathbb{R}$ is bounded.
3. Every continuous function $f : [0,1] \to \mathbb{R}$ is uniformly continuous.
4. Gödel completeness theorem.
5. Every ring has a prime ideal.
6. Brower's fixed point: every uniformly continuous function $\varphi : [0,1]^n \to [0,1]^n$ has a fixed point.

$\mathsf{ACA}_0 = \mathsf{WKL}_0 +$ comprehension for arithmetic formulae.

$\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \Leftrightarrow \tau$, where τ could be:

1. Every bounded sequence of real numbers has a least upper bound.
2. Bolzano-Weirstrass theorem.
3. Every commutative ring has a maximal ideal.
4. Every vector space has a basis.
5. Every abelian group has a trascendence basis.

$\mathsf{ATR}_0 = \mathsf{ACA}_0 + \Sigma^1_1$-separation.

$\mathsf{RCA}_0 \vdash \mathsf{ATR}_0 \Leftrightarrow \tau$, where τ could be:

1. Any two countable well orderings are comparable.
2. Lusin's separation theorem: any two disjoint analytic sets can be separated by a Borel set.
3. Every open subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

Π^1_1-CA_0 $=$ $\mathsf{ATR}_0$$+$ Π^1_1-comprehension.

$\mathsf{RCA}_0 \vdash \Pi^1_1$-$\mathsf{CA}_0 \Leftrightarrow \tau$, where τ could be:

1. Every difference of two open sets in the Baire space $\mathbb{N}^{\mathbb{N}}$ is determined.
2. Every abelian group is the direct sum of a divisible group and a reduced group.
3. Every tree has a largest perfect subtree.
4. Every closed subset of $\mathbb{R}$ is the union of a countable set and a perfect set.

To be honest I must point out two main points that I haven't specified so far.

1. Not all mathematics! Sets of sets of natural numbers are not part of second order arithmetic, hence it cannot handle essentially uncountable mathematics.
2. Some theorems are placed in intermediate subsystems.

MAIN REFERENCE

Alberto Marcone (Udine).

MAIN TEXTBOOK

Stephen G. Simpson,
Subsystems of second order arithmetic. Cambridge University Press, 2nd ed., 2009.

My personal advice:
Alberto Marcone, Equivalenze tra teoremi: il programma di ricerca della reverse mathematics.
La matematica nella società e nella cultura, Rivista dell'Unione Matematica Italiana 2 (2009), 101-126.

THANKS FOR THE ATTENTION