Joint Distributions and Copulas: Ideas and Applications

Stochastic models

- Forecast of future events is one of the oldest human dreams;
- The ancient Greeks used signs to predict the future
- Astronomical mathematical models are used for years to predict the motion of the stars
- Other mathematical models allow predictions in different branches of science

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Remark

Many phenomena are not deterministic: they are subject to random evolution.
However they show some regularities and previsions are still possible.
Of course these previsions cannot be deterministic!

Stochastic models

Remark

Two random variables are characterized by their joint ditribution: the observation of one of the two r.v. is a *scientific* sign to argue values of the other (in a probabilistic framework)!

Single variables and random vectors... dependences

Single random variables exhibit regularities: mean value, variability, distribution. We can predict their value in advance when we know their distribution

Random Vectors

- The knowledge of one component may change our previsions on the other component;

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Random Vectors

- The knowledge of one component may change our previsions on the other component;
- Joint tail event may become more probable than tail events of a single component (extreme downside events may occur simoultaneously: Chernobyl accident; global financial crisis of 2008-2009).

Models and Random Variables

Phenomenon:

- Waiting time at the bus stop
- Weight of 7 years old children
- Intertime between two eruptions
- Value of an option on July 25
- ....

These quantities are random but they exhibit specific random regularities.

Their model is a random variable whose law captures their random regularities.

One Dimensional Random Variables

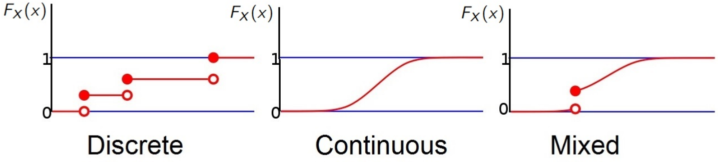
Random Variables are described through their cumulative distribution
One dimensional case:

$$F_X(x) = P(X \leq x)$$

Properties

1. The probability that X lies in the semi-closed interval $(a, b]$, where $a < b$ is $P(a < X \leq b) = F_X(b) - F_X(a)$
2. Cumulative Distribution Function F_X is non-decreasing
3. Cumulative Distribution Function F_X is right-continuous, which makes it a cadlag function;
4. $\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to +\infty} F(x) = 1.$

One Dimensional Random Variables



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Remark

Every function with these four properties is a CDF: a random variable can be defined such that the function is the cumulative distribution function of that random variable.

Cumulative Distributions/Probability Densities

Continuos Random Variables admit Probability Density Function $f_X(x)$

Properties

1. $f_X(x) \geq 0$
2. Cumulative Distribution Function F_X is non-decreasing
3. $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

Remark

It is easy to determine new Probability Density Functions Distributions

Remark

It is easy to determine new Cumulative Distributions

More Complex Phenomena

Phenomenon:

- Waiting time and time interval of my arrival at the bus stop;
- Weight and Height of 7 years old children;
- Intertime between two eruptions and lenght in time of the eruption;
- Value of an option on July 25 and its value on June 30;
- ....

These quantities are random, they exhibit specific random regularities.

Remark

These quanties are related: the knowledge of one of them improves our knowledge of the other.

Two Dimensional Random Variables

Random Variables are described through their cumulative distribution
Two dimensional case:

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

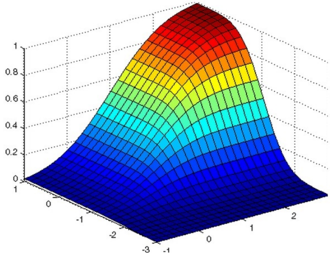
Properties

1. $F_{X,Y}(x,y)$ is $2-$ increasing, i.e.

   $$V_H(B) = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1) \geq 0$$

   where $B = (x_1,x_2] \times (y_1,y_2]$;
2. Right-continuous for each of its variables;
3. $0 \leq F_{X,Y}(x,y) \leq 1$;
4. $\lim_{x\to-\infty} F_{X,Y}(x,y) = 0$, $\lim_{y\to-\infty} F_{X,Y}(x,y) = 0$;
   $\lim_{x\to\infty} F_{X,Y}(x,y) = F_Y(y)$; $\lim_{y\to\infty} F_{X,Y}(x,y) = F_X(x)$;
   $\lim_{\substack{x\to\infty \\ y\to\infty}} F_{X,Y}(x,y) = 1$.

Bivariate Cumulative Distributions



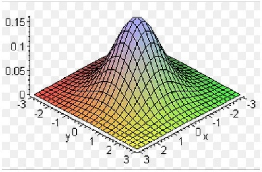
Remark

Every function with these four properties is a bivariate CDF: a random variable can be defined such that the function is the cumulative distribution function of that random variable.

Bivariate Distributions

When the random variable is continuos we can introduce its probability density function. Bivariate distributions are known in avery limited number of instances:

- Bivariate Normal distribution;
- Bivariate Student Distribution;
- Multinomial Distribution;



Remark

It is not easy to determine bivariate cumulative distributions

Dependent Random Variables

- Joint distribution captures dependences between random variables but its shape is influenced by the marginal behaviour of each component
- Dependence can be captured through specific indexes:
  - Covariance

    $$cov\left(X, Y\right) = E$$

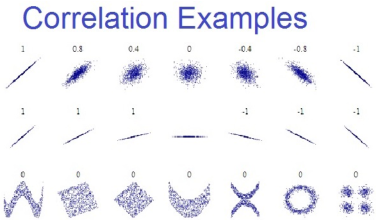
  - Correlation Coefficient

    $$\rho_{X,Y} = \frac{cov\left(X, Y\right)}{\sigma_X \sigma_Y} = \frac{E\left[\left(X - \mu_X\right)\left(Y - \mu_y\right)\right]}{\sigma_X \sigma_Y}$$

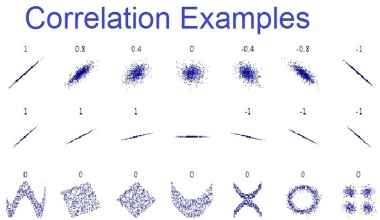
  - Kendall τ index
    $\tau_{X,Y} = P\left[\left(X_1 - X_2\right)\left(Y_1 - Y_2\right) > 0\right] - P\left[\left(X_1 - X_2\right)\left(Y_1 - Y_2\right) < 0\right]$
    where (X_1, Y_1) and (X_2, Y_2) are i.i.d. random variables
  - Mutual Information

    $$I\left(X, Y\right) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f_{X,Y}\left(x, y\right) \log \frac{f_{X,Y}\left(x, y\right)}{f_X(x) f_Y\left(y\right)} dy$$

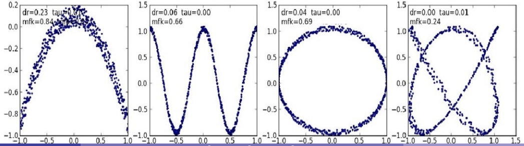
Different Indexes - Different Detected Features



Different Indexes - Different Detected Features



ρ versus τ versus Mutual Information



Index versus Joint Distribution

Remark

- Indexes summarize the joint behavior of two random variables but they lose an important part of the information. each index has its advantages and its shortcomings;
- the knowledge of the joint distribution gives more complete information on the random variables but it merges joint and marginal behaviors.

Index versus Joint Distribution

Remark

- Indexes summarize the joint behavior of two random variables but they lose an important part of the information. each index has its advantages and its shortcomings;
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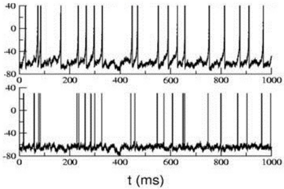
Neuron Spike Trains



Simultaneous spikes reaveal a dependence between the neurons or they are due to the chance?

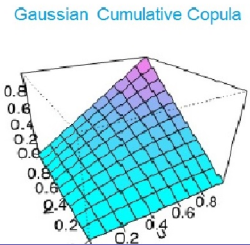
Copulas

Idea

Consider the three values:

1. $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$
2. $F_X(x) = P(X \leq x)$
3. $F_Y(y) = P(Y \leq y)$

Each of them belongs to the interval $(0, 1)$. Plot these value in a cube of unitary side



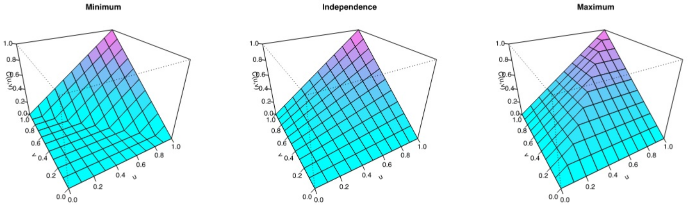
Copulas

Definition

A two-dimensional copula is a function $C : [0,1]^2 \to [0,1]$ with the following properties:

1. $C(u;0) = C(0;v) = 0$ and $C(u;1) = u, C(1;v) = v$ for every $u, v \in [0;1]$;
2. C is 2-increasing, i.e. for every $u_1, u_2, v_1, v_2 \in [0;1]$ such that $u_1 \leq u_2, v_1 \leq v_2$,

$$C(u_1, v_1) + C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) \geq 0$$



Sklar's Theorem

Theorem

Let F_1 and F_2 be two univariate distributions. It comes that $C(F_1(x_1), F_2(x_2))$ defines a bivariate probability distribution with margins F_1 and F_2.

Theorem

Let $F_{1,2}$ be a two-dimensional distribution function with margins F_1 and F_2. Then $F_{1,2}$ has a copula representation:

$$F_{1,2}(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

The copula C is unique if the margins are continuous.

Copulas

Remark

Let $U = F_X(x), V = F_Y(y)$. The random variables U and V are uniform.
Proof
$P(U \leq u) = P(F_X(x) \leq u) = P(X \leq F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u, 0 \leq u \leq 1$

Copulas

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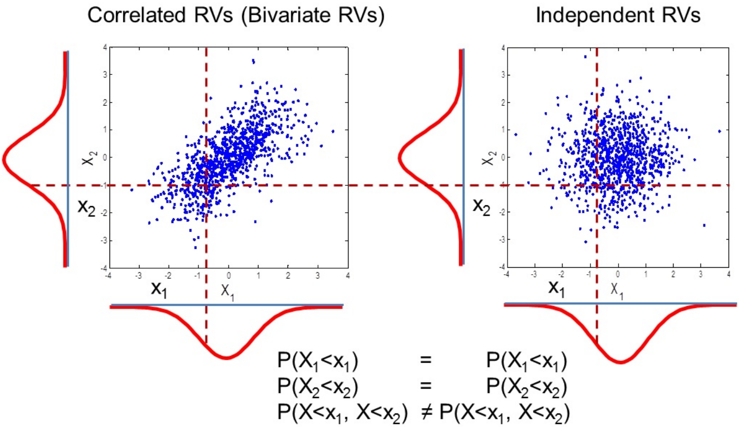
Remark

Copulas can be read as the Joint Cumulative Distribution of couples of Uniform Random Variables: $C(u, v) = P(U \leq u, V \leq v)$
The same Copula corresponds to different Joint Distributions. These Joint Distribuions are obtained computing the copula with different Marginals.

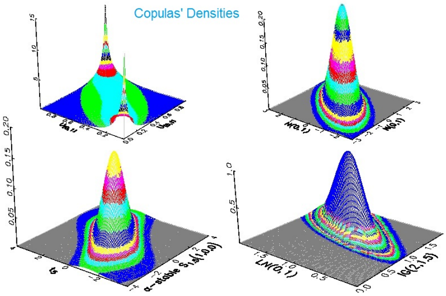
Same Marginals but different Joint Distribution



Remark

Same Copula but different Marginals: different Joint Distribution. We can construct new joint distributions!

Some Copula Families have Densities



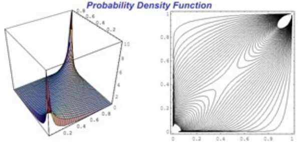
Examples of Copulas

name	bivariate copula $C_\theta(u,v)$	parameter θ
Clayton	$\left(\max\left\{u^{-\theta}+v^{-\theta}-1;0\right\}\right)^{-1/\theta}$	$\theta \in [-1,\infty)\backslash\{0\}$
Ali-Mikhail-Haq	$\frac{uv}{1-\theta(1-u)(1-v)}$	$\theta \in [-1,1)$
Gumbel	$\exp\left(-\left((-\log(u))^\theta+(-\log(v))^\theta\right)^{1/\theta}\right)$	$\theta \in [1,\infty)$
Frank	$-\frac{1}{\theta}\log\left(1+\frac{(\exp(-\theta u)-1)(\exp(-\theta v)-1)}{\exp(-\theta)-1}\right)$	$\theta \in \mathbb{R}\backslash\{0\}$
Joe	$1-\left((1-u)^\theta+(1-v)^\theta-(1-u)^\theta(1-v)^\theta\right)^{1/\theta}$	$\theta \in [1,\infty)$
Independence	uv	

Gumbel

$$C(u,v)=\exp\left(-\left[(-\ln u)^a+(-\ln v)^a\right]^{\frac{1}{a}}\right)$$

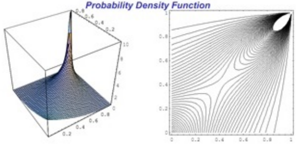
$$\varphi(u)=(-\ln u)^a,\quad a\geq 1 \qquad \tau_a=1-1/a$$



HRT

$$C(u,v)=u+v-1+\left((1-u)^{-a}+(1-v)^{-a}-1\right)^{-\frac{1}{a}},\quad a>0$$

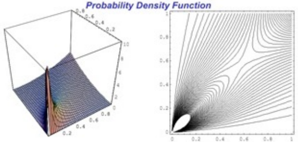
$$\tau_a=\frac{a}{a+2}$$



Clayton

$$C(u,v)=(u^{-a}+v^{-a}-1)^{-\frac{1}{a}}$$

$$\varphi(u)=\frac{(u^{-a}-1)}{a},\quad a>0 \qquad \tau_a=\frac{a}{a+2}$$



Important Features of Copulas

Theorem

Let X and Y be continuos random variables with Copula $C_{X,Y}$ If α and β are strictly increasing on $RanX$ and $RanY$ respectively, then $C_{\alpha(X),\beta(Y)} = C_{X,Y}$. Thus $C_{X,Y}$ is invariant under strictly increasing transformations of X and Y

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Theorem

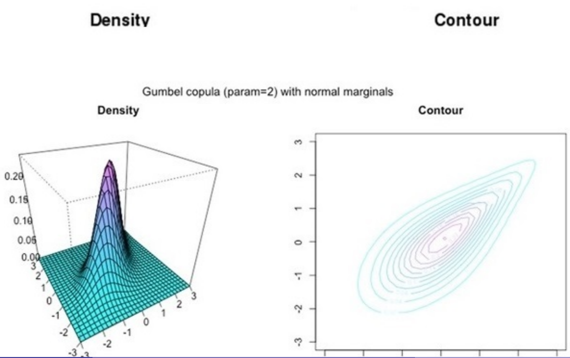
Let X and Y be continuos random variables with Copula $C_{X,Y}$ Let α and β be strictly monotone on $RanX$ and $RanY$ respectively.

- *If α is strictly increasing and β is strictly decreasing, then* $C_{\alpha(X),\beta(Y)}u, v) = u - C_{X,Y}(u, 1 - v)$
- *If α is strictly decreasing and β is strictly increasing, then* $C_{\alpha(X),\beta(Y)}u, v) = v - C_{X,Y}(1 - u, v)$
- *If α and β are both strictly decreasing, then* $C_{\alpha(X),\beta(Y)}u, v) = u + v - 1 + C_{X,Y}(1 - u, 1 - v)$

New Joint Distributions

Idea

1. Select a Copula having the dependence feature of interest (there are many families of Copulas, each depending from parameters that may change their shape)
2. Apply Marginals of interest to get the new Joint Ditribution



Empirical Copulas

1. Consider the sample $\left(X^i, Y^i\right)$ $i = 1, ..., n$ from a vector (X, Y) with continuous marginals.
2. The corresponding observations for the copula are:
$$\left(U^i, V^i\right) = \left(F_X\left(X^i\right).F_Y\left(Y^i\right)\right), i = 1, ..., n$$
3. The Marginal Distributions $F_X(x)$ and $F_Y(y)$ are unknown: we substitute them with the empirical Distribution Functions:
$$\begin{aligned} F_X^n(x) &= \frac{1}{n}\sum_{i=1}^{n} 1\left(X^i \leq x\right) \\ F_Y^n(y) &= \frac{1}{n}\sum_{i=1}^{n} 1\left(Y^i \leq x\right) \end{aligned}$$
then the observations of the copula become
$$\left(\widetilde{U}^i, \widetilde{V}^i\right) = \left(F_X^n\left(X^i\right).F_Y^n\left(Y^i\right)\right), i = 1, ..., n$$
4. The corresponding empirical Copula is defined as:
$$C^n(u, v) = \frac{1}{n}\sum_{i=1}^{n} 1\left(\widetilde{U}^i \leq u, \widetilde{V}^i \leq v\right)$$

Applications

Some subjects modeled through copulas

- Foreign exchange distributions: joint behavior of euro-dollar ...
- Mineral resource estimation: joint presence of specific minerals
- Reliability problems:joint crash of mechnical parts
- Actuary: incidence of two individuals die and corresponding annual insurance premium
- Neuroscience: joint behaviour of two neurons in a network
- Epidemiology: joint evolution of illness

Further topics on copulas

- Extension to higher dimensions
- Simulation of Copulas
- Copulas for Stochastic Processes
- New Classes of Copulas
- ....

Thank you!