Finite and Infinite Ramsey Theorem

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How many people do you need to invite in a party in order to have that either n of them mutually know each other or n of them mutually do not know each other?

If you have 6 people at a party then either 3 of them mutually know each other or 3 of them mutually do not know each other.

If you have 18 people at a party then either 4 of them mutually know each other or 4 of them mutually do not know each other.

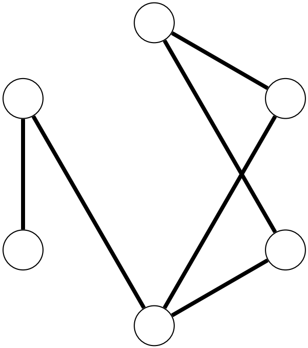
How many people do you need to invite in a party in order to have that either n of them mutually know each other or n of them mutually do not know each other?

How may we know that such number exists for any n?

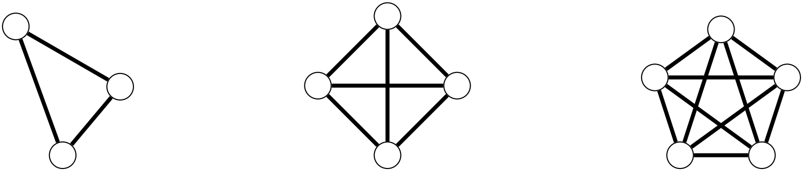
Thanks to F.P. Ramsey!

Graphs

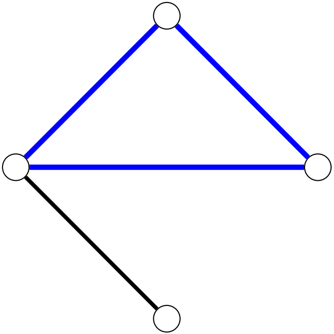
A graph is an ordered pair $G = (V, E)$ composed by a set V of nodes together with a set E of edges, which are 2-elements subsets of V.

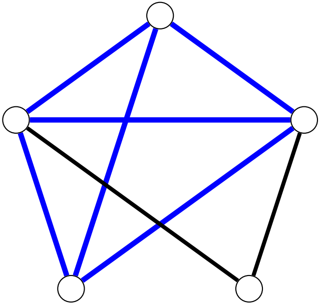


A graph is complete if for each pair of nodes there is an edge connecting them. For each $n \in \mathbb{N}$, K_n is the complete graph with n nodes.



A clique in a graph is a subset of its nodes such that every two nodes in the subset are connected by an edge.

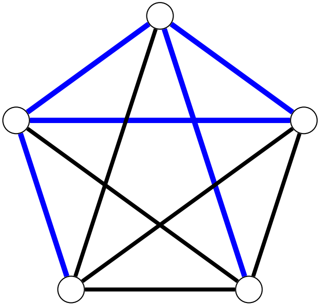




Let $r \in \mathbb{N}$. A coloring of the edges of a graph in r colors is a function

$$c : E \to r.$$

An edge coloring with r colors is a partition of the edge set into r classes.



Finite Ramsey Theorem

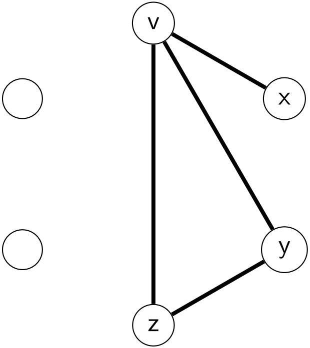
Theorem (Finite Ramsey Theorem for pairs in two colors)

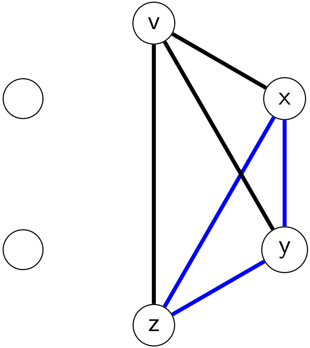
For any $n, \ m \in \mathbb{N}$ there exists $t \in \mathbb{N}$ such that: for any coloring in 2 colors of the edges of the complete graph with t nodes there exists a n-clique homogeneous in color 0 or a m-clique homogeneous in color 1.

A homogeneous set is a subset of the vertices such that each edge has the same color.

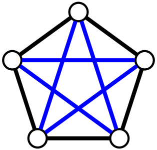
Example

If you have 6 people at a party then either 3 of them mutually know each other or 3 of them mutually do not know each other.





6 is the minimum number n for which if you have n people at a party then either 3 of them mutually know each other or 3 of them mutually do not know each other. In fact we may find a coloring on K_5 without any monochromatic triangle.



Definition

Let $n, m \in \mathbb{N}$, $R(n, m)$ is the minimum $t \in \mathbb{N}$ such that for any coloring on the complete graph on K_t there exists either a n-clique homogeneous in color 0 or a m-clique homogeneous in color 1.

It is an open problem to determine the values of $R(n, m)$ for most values of n and m.

R	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8
3	1	3	6	9	14	18	23	28
4	1	4	9	18	25	$[35, 41]$	$[49, 61]$	$[56, 84]$
5	1	5	14	25	$[43, 49]$	$[58, 87]$	$[80, 143]$	$[101, 216]$
6	1	6	18	$[35, 41]$	$[58, 87]$	$[102, 165]$	$[113, 298]$	$[127, 495]$
7	1	7	23	$[49, 61]$	$[80, 143]$	$[113, 298]$	$[205, 540]$	$[216, 1031]$
8	1	8	28	$[56, 84]$	$[101, 216]$	$[127, 495]$	$[216, 1031]$	$[282, 1870]$
9	1	9	36	$[73, 115]$	$[125, 316]$	$[169, 780]$	$[233, 1713]$	$[317, 3583]$

Theorem

$$\forall n \in \mathbb{N} \setminus \{0\}\ \forall m \in \mathbb{N} \setminus \{0\}\ (R(n+1, m+1) \leq R(n, m+1) + R(n+1, m)).$$

Proof.

Given a coloring in two colors of the complete graph on

$$R(n, m+1) + R(n+1, m)$$

many nodes, take a node x. There are

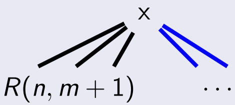
$$R(n, m+1) + R(n+1, m) - 1$$

many edges from x. Then it has either $R(n, m+1)$ many 0-edges or $R(n+1, m)$ many 1-edges.

Proof.

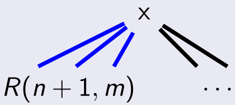
Case 1.

Let consider the graph induced by the $R(n, m+1)$ nodes connected with color 0 with x. If there exists a n-clique in color 0, then by adding x we obtain an homogeneous $n+1$-clique in color 0. Otherwise we have a m-clique in color 1 and we are done.



Proof.

Case 2.
Let consider the graph induced by the $R(n+1, m)$ nodes connected with color 1 with x. If there exists a m-clique in color 1, then by adding x we obtain an homogeneous $m+1$-clique in color 1. Otherwise we have a $n+1$-clique in color 0 and we are done.



□

Infinite Ramsey Theorem

If you have $\mathbb{N}$ people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

Theorem (Infinite Ramsey Theorem for pairs)

Let $K_{\mathbb{N}}$ be the complete graph on $\mathbb{N}$ nodes. For any $n \in \mathbb{N}$ and for every n-coloring on $K_{\mathbb{N}}$, there exists an infinite homogeneous set.

Complete disorder is impossible

Theodore Samuel Motzkin

Applications

Theorem (Schur)

For any partition of the positive integers into a finite number of parts, one of the parts contains three integers $x, \ y, \ z$ *such that*

$$x + y = z.$$

Proof.

Let

$$p : \mathbb{N} \to r$$

be a partition into r classes. Let us define an assignment of r colors

$$c : [\mathbb{N}]^2 \to r$$

such that $c(\{x, y\}) = m$ if and only if $p(|x - y|) = m$.

Proof.

Thanks to RT^2_r we have a monochromatic triangle: i.e. there exist $i > j > k$ such that

$$p(|i-j|) = p(|j-k|) = p(|i-k|).$$

So, by defining

$$\begin{aligned} x &= i - j \\ y &= j - k \\ z &= i - k, \end{aligned}$$

we have

$$x + y = (i-j) + (j-k) = (i-k) = z.$$

□

Theorem

Any infinite linear order $\prec$ contains either an increasing infinite chain or a decreasing infinite chain.

Proof.

Let c be the following coloring: for each $x < y \in \mathbb{N}$

$$c(\{x, y\}) = \begin{cases} 0 \text{ iff } x \prec y \\ 1 \text{ iff } x \succ y. \end{cases}$$

Thanks to Infinite Ramsey Theorem, there exists an infinite homogeneous set. If there is an homogeneous set in color 0 we obtain an infinite increasing chain. Otherwise, if the homogeneous set is in color 1 we obtain an infinite decreasing chain. □

Theorem (AC)

Let $K_{\mathbb{R}}$ be the complete graph on $\mathbb{R}$. There exists a 2-coloring of $K_{\mathbb{R}}$ for which there are no homogeneous sets of size $|\mathbb{R}|$.

Proof.

Let $\lhd$ a well ordering of $\mathbb{R}$ and let c be the following 2-coloring of $K_{\mathbb{R}}$.
For any $x \lhd y$

$$c(\{x, y\}) = \begin{cases} 0 \text{ if } x < y \\ 1 \text{ otherwise} \end{cases}$$

Suppose by contradiction that there is an homogeneous set of size $|\mathbb{R}|$.
Then we obtain a decreasing or increasing sequence of $|\mathbb{R}|$ many reals.
This is a contradiction since $\mathbb{R}$ is separable. □

Thanks.