Orbits and space-time symmetries in the classical the N-body problem

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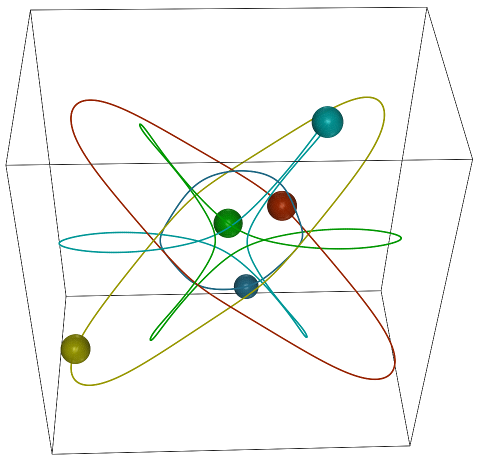


Curiosando nella Matematica - Browsing through Mathematics
Università di Torino
April 28th 2014

1 The equations of Celestial Mechanics

From the mathematical point of view, the motion of a systems of mass point particles is governed by a system of differential equations in Newtonian form. Let us denote by $x_i(t)$ the position of the i-th particle in the space, the acceleration of this particle is given by the derivative in time of the velocity $v(t) = \dot{x}_i(t)$, and, finally, by the second derivative of the position $x(t)$ with respect to the time variable $a(t) = \ddot{x}_i(t)$:

$$\underbrace{m_i \ddot{x}_i}_{\text{mass} \times \text{acceleration}} = \underbrace{F_i}_{\text{total force}}, \qquad i = 1, \ldots, n$$

The motion of the planets and the other celestial bodies is governed by Newton's law of universal gravitation, which says that the gravitational force exerted on one particle by another one is directed along the vector joining the two and it intensity is proportional to the inverse of the square of their distance and the product of the masses.

Consequently, the total force acting on one particle is the sum of all the attraction forces exerted by the others

$$F_i = \sum_{j \neq i} F_{ij} = \sum_{j \neq i} m_i m_j \frac{x_j - x_i}{\|x_j - x_i\|}, \qquad i = 1, \ldots, n$$

2 The two body problem

$$\begin{cases} m_1\ddot{x}_1 = Gm_1m_2\dfrac{x_2 - x_1}{\|x_2 - x_1\|^3}, \\ m_2\ddot{x}_2 = Gm_2m_1\dfrac{x_1 - x_2}{\|x_1 - x_2\|^3} \end{cases}$$

We notice that, when summing the two equations, the right hand side vanishes:

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0$$

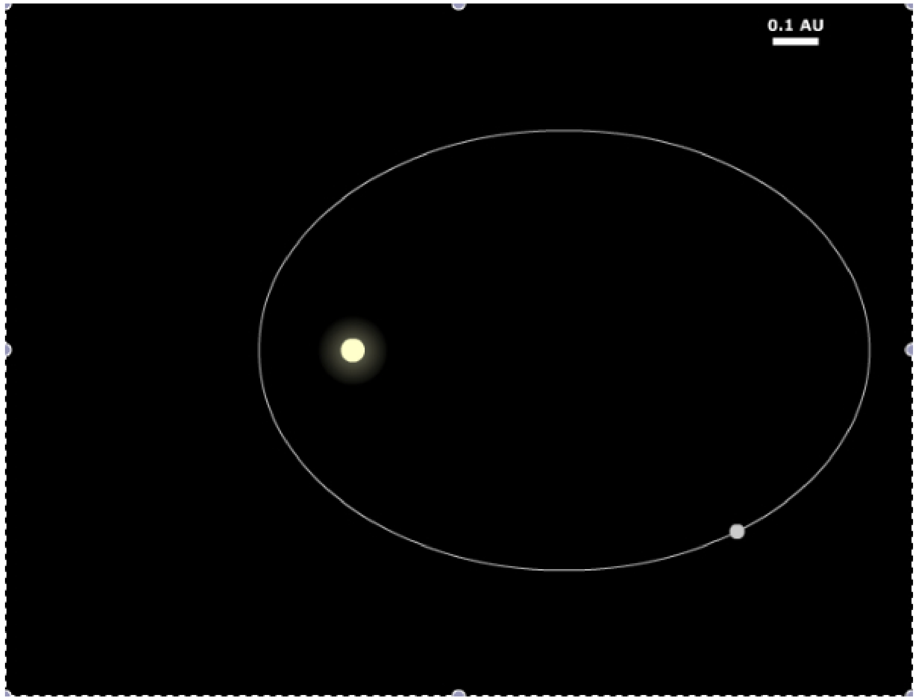
this is the center of mass conservation law. It says that the barycenter moves as there were no forces acting on it, and therefore it undergoes a uniform rectilinear motion.

Thanks to this conservation law, the system of two differential equations reduces to that of the motion of one unique body subject to an attractive central force, whose intensity is always proportional to the inverse square of the distance to the center of attraction (which remains at rest).

3 Central force: Kepler's laws

Newton deduced the inverse square law starting from the three Kepler's laws of the planets motion:

- the planet moves on an elliptic orbit, the Sun occupies one of the foci .
- The planet-Sun ray sweeps equal areas in equal times.
- The square of the planet's revolution periods are proportional to the cube of the major semiaxis of their orbit.

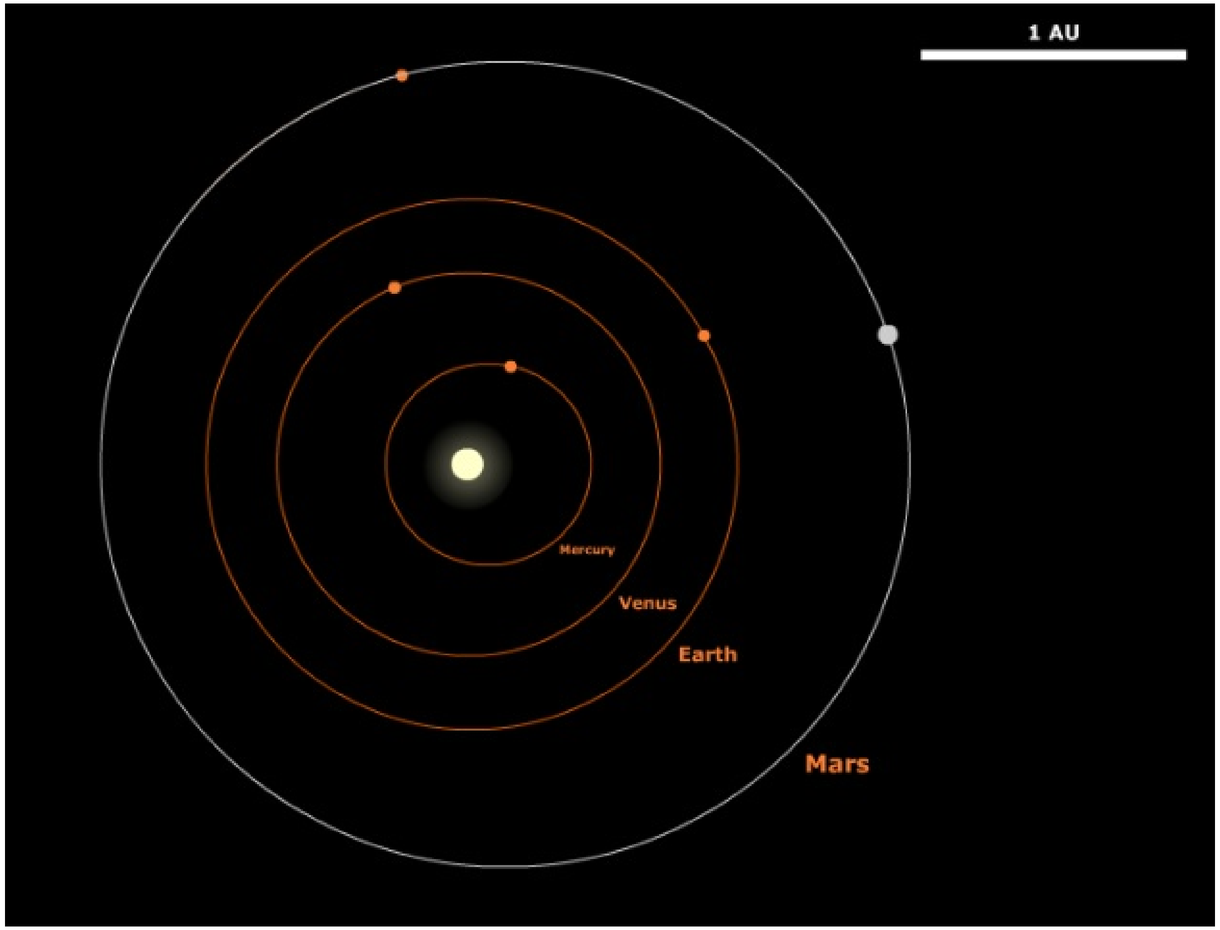


Let us notice that the force is infinite at the attraction center. A long standing question is whether one can extend the motion after a collision and under which rules. Tullio Levi-Civita (1873-1941) proposed a space-time transformation which regularizes binary collisions of the three body problem.

- The Levi-Civta regularization transforms the one-center problem into an harmonic oscillator!

4 The solar system

The solar system counts many celestial bodies. As the masses of the planets are much smaller than that of the Sun, in a first approximation, we can neglect the mutual attraction forces between the planets; then, each planet will move independently of the others:



Within this approximation, we can see the problem as a perturbation of a system of harmonic oscillators

➡ The theory of perturbations.

5 The three body problem

$$\begin{cases} m_1\ddot{x}_1 = Gm_1m_2\dfrac{x_2 - x_1}{\|x_2 - x_1\|^3} + Gm_1m_3\dfrac{x_3 - x_1}{\|x_3 - x_1\|^3} \\ m_2\ddot{x}_2 = Gm_2m_1\dfrac{x_1 - x_2}{\|x_1 - x_2\|^3} + Gm_2m_3\dfrac{x_3 - x_2}{\|x_3 - x_2\|^3} \\ m_3\ddot{x}_3 = Gm_3m_1\dfrac{x_1 - x_3}{\|x_1 - x_3\|^3} + Gm_3m_2\dfrac{x_2 - x_3}{\|x_2 - x_3\|^3} \end{cases}$$

where m_i are the (positive) masses and $x_i(t)$, $i = 1, 2, 3$ are three dimensional vectors, depending on the time t. As we have already seen, the two body problem can be solved by the use of Levi–Civita transformations. It transforms into a harmonic oscillator: the two body problem is integrable.

On the other hand, the three body problem can not be solved and can not be seen, in its full generality, as a perturbation of a simple integrable problem. A key to catch this complexity rests in the study of some relatively simple solutions, which repeats their motions after a certain time: the periodic orbits.

6 Orbits of the N-body problem

The system of differential equations of the N-body problem is as follows:

$$m_i \ddot{x}_i = G \sum_{\substack{j \neq i \\ j=1}}^{N} m_i m_j \frac{x_j - x_i}{\|x_j - x_i\|^3} \qquad i = 1, \ldots, N \qquad (*)$$

A solutions, or trajectory, or orbit of the system is a vector valued function $(x_1(t), \ldots, x_N(t))$ of twice differentiable functions which verify $(*)$ at each time $t \in (a, b)$. To be meaningful, we have to require $x_i(t) \neq x_j(t)$ for every $t \in (a, b)$. This requirement prevents collisions among the bodies.

Concerning the time behavior, a trajetory can be

- periodic, if there exists $T > 0$ such that $x_i(t + T) = x_i(t)$ for every $t \in \mathbb{R}$ and $i = 1; \ldots, N$. Esempio: $x_i(t) = \cos t$.
- quasi-periodic, if there are functions $Y_i(\xi_1, \ldots, \xi_k)$, periodic in each variable ξ_i such that $x_i(t) = Y_i(t, t, \ldots, t)$, for every i. Example: $x_i(t) = \cos t + \cos \sqrt{2}t$; here $Y_i(t_1, t_2) = \cos t_1 + \cos \sqrt{2}t_2$.
- almost-periodic, if, for every $\varepsilon > 0$, there exists $\tau \in \mathbb{R}$ such that, for every t and i, $\|x_i(t) - x_i(t + \tau)\| < \varepsilon$. Example, $\sum_{n=1}^{+\infty} 2^{-n} \cos(t/n)$.

7 Periodic orbits

The mathematician who first understood the role of periodic orbits in the comprehension of the full dynamics of the N-body problem was the french Henri Poincaré (1854-1912):

"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputée inabordable...".

According with Poincaré, periodic trajectories are dense in the phase p,ane:

"...voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans[1] *le n. 13 et une solution particulière quelconque de ces équations, one peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut."*

Even today, this statement has not found a rigorous proof and remains the Poincaré periodic points conjecture. Since the times of Poincaré, it has became part of the mathematical way of thinking that periodic orbits, beside being interesting by their own, may capture other trajectories for long times.

The idea is that one could travel through the phase space by following (shadowing) a concatenation of periodic orbits. One could jump from one periodic orbit to another as buses.

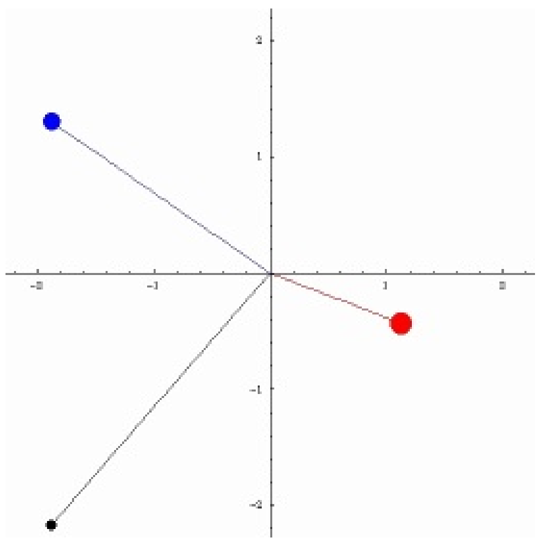
➡ Problem: how to find enough periodic solutions to capture the complexity of the system.

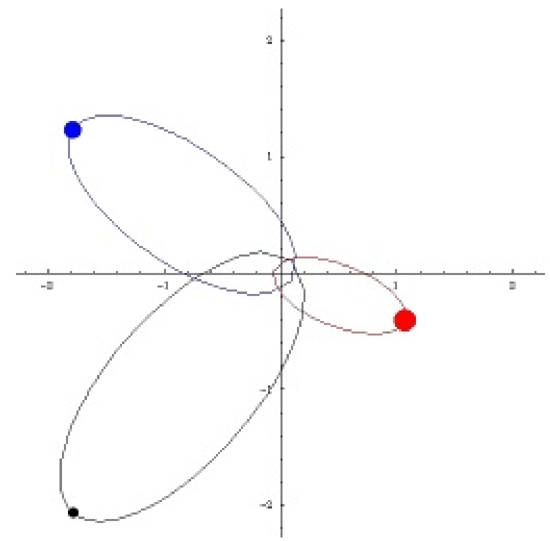
[1] La formula n. 13 menzionata da Poincaré è l'equazione di Hamilton.

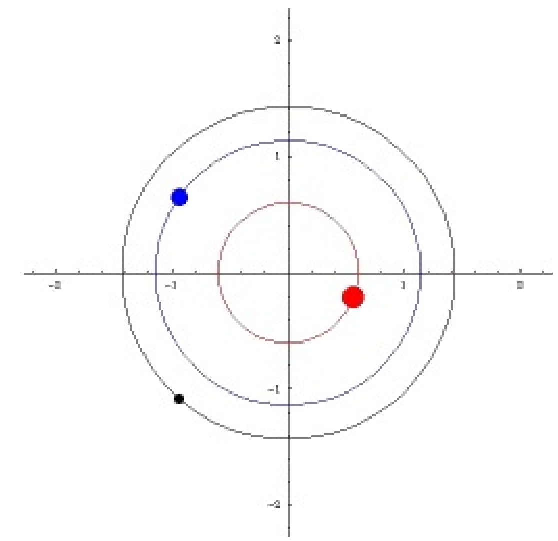
8 Homographic motions and centrac configurations

The simplest periodic solutions are associated with central configurations; their main feature is that they keep a constant shape (up to rotations and dialations) which rotates and expands and shrinks in time.

As the shape remains unchanged, these trajectories are called homographic. In such particular motions, each body moves under the effect of one single center of attraction, located in the barycenter, hence describing an ellipse (or a parabola, or a hyperbole).



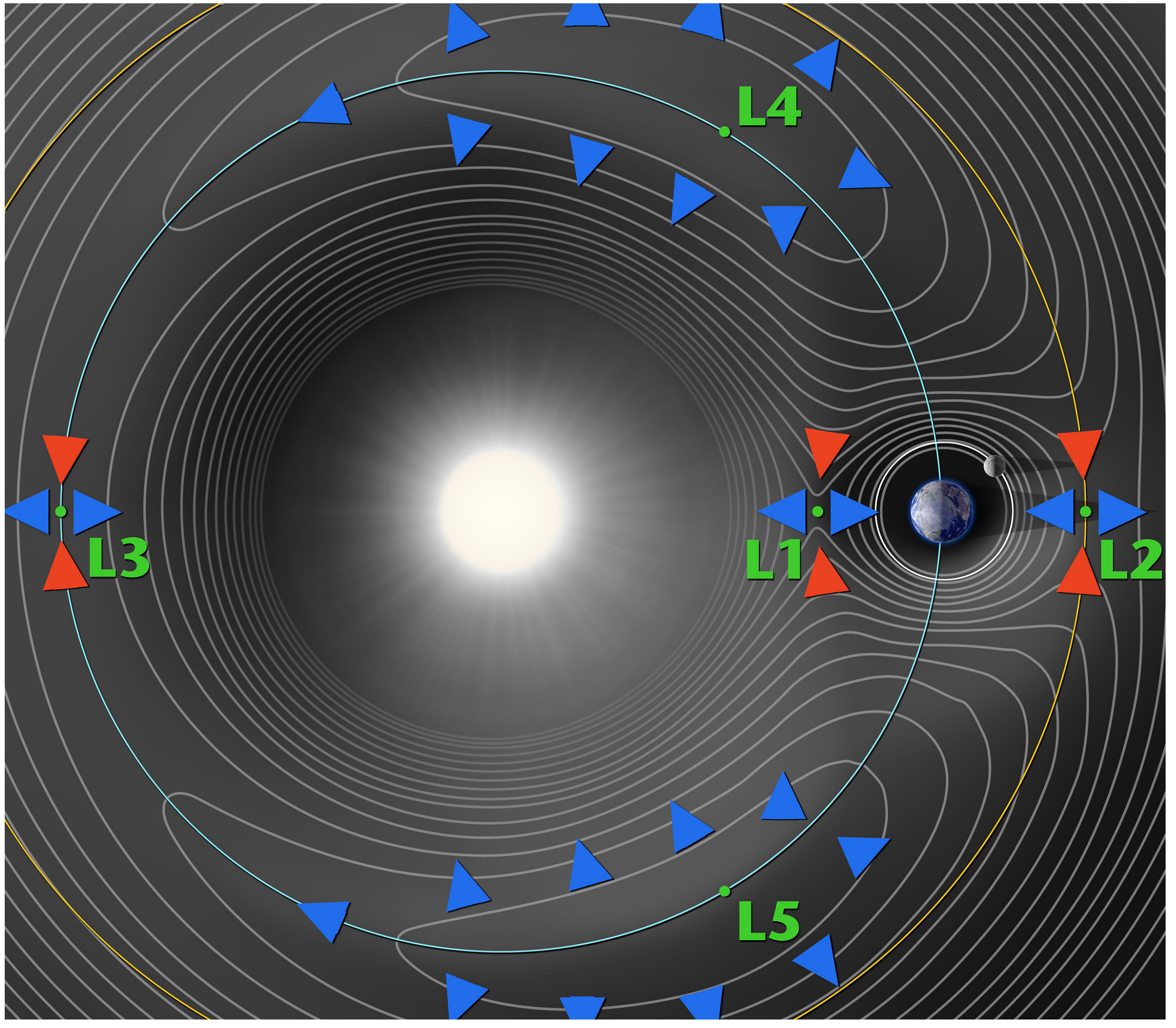




A central configuration has the property that the force resulting on a one body is proportional to the vector joining it to the barycenter. In the planar case they are equilibria in a rotating frame.

9 Lagrangian points. The restricted three body problem

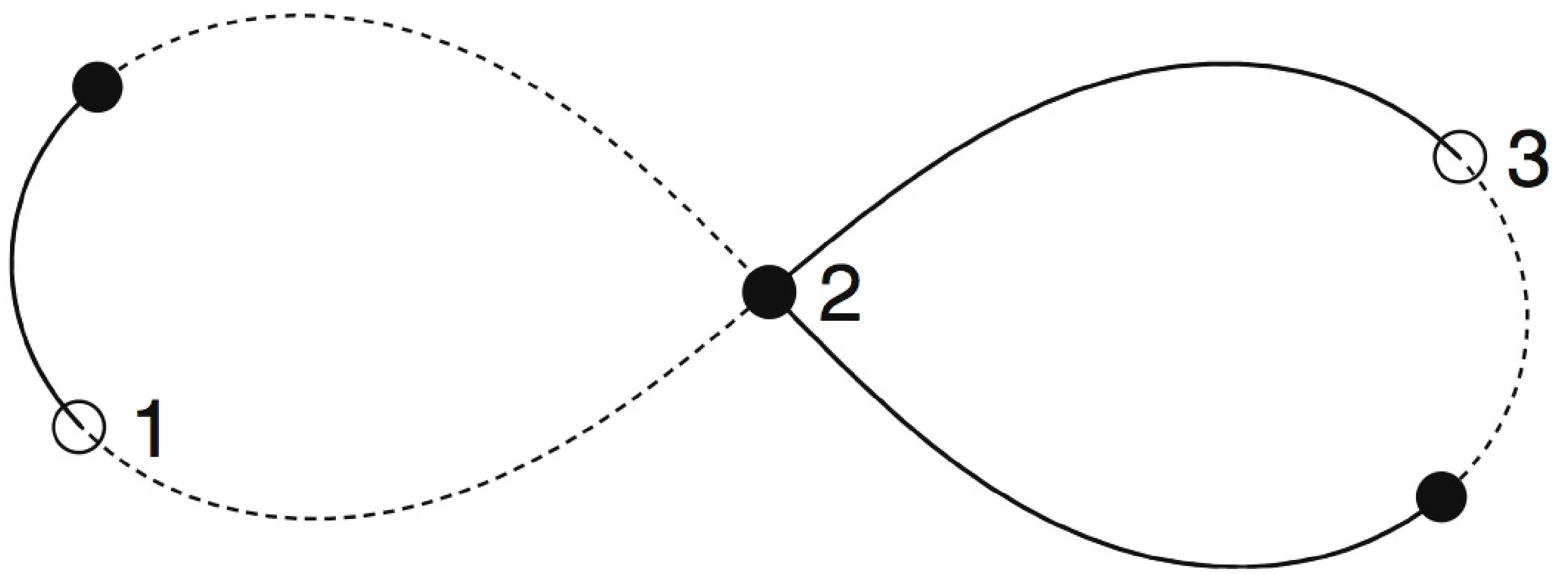
In the circular restricted three-body problem, it is assumed that there are two major bodies moving on a circle (sometimes a ellipse) of the two-body problem, while the third, much smaller, moves under the action of the two major ones. In a reference frame moving with the two larger bodies, in the circular case, we find is a problem with two centers of attraction and centripetal force. There are five equilibrium positions, which were identified as stationary points of the effective potential by J.L. Lagrange (1736-1813).



There are several ways to find periodic orbits. One possibility, which has been widely explored in the past, is to start with a simpler problem, whose periodic solutions are well known, and then perturbing it by the introduction of a new body, having a very small mass, or being very faraway from the previous. This falls again in the theory of perturbations.

In recent years, many new periodic orbits orbits have been discovered, using the symmetries (in space and time) and the least action principle.

In 2000, two mathematicians (one French and the other American), Alain Chenciner and Richard Montgomery used the least action principle (Lagrange) with symmetries to find a surprising periodic orbit for the three bodies:



11 Symmetries of the eight

The simplest way to define the symmetry group which gives rise to the eight is the following. As usual,

$$x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^6$$

denote the positions of the three bodies having all the same mass.

The group is generated by the following two space-time reflections: : the first is

$$\boxed{x_1(-t) = -x_3(t)\ ,\ x_2(-t) = -x_2(t)\ ,\ x_3(-t) = -x_1(t)\ .}$$

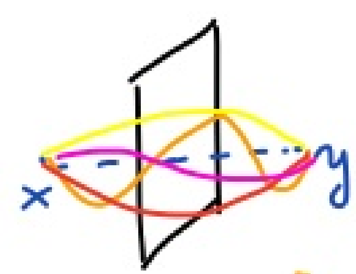
Notice that, at time $t = 0$ il the second body is at the origin while the first and third are opposite to the origin (collinear configuration). The second symmetry is similar, bur with exchanged roles:

$$\boxed{x_1(1-t) = -x_2(t)\ ,\ x_2(1-t) = -x_1(t)\ ,\ x_3(1-t) = -x_3(t)}$$

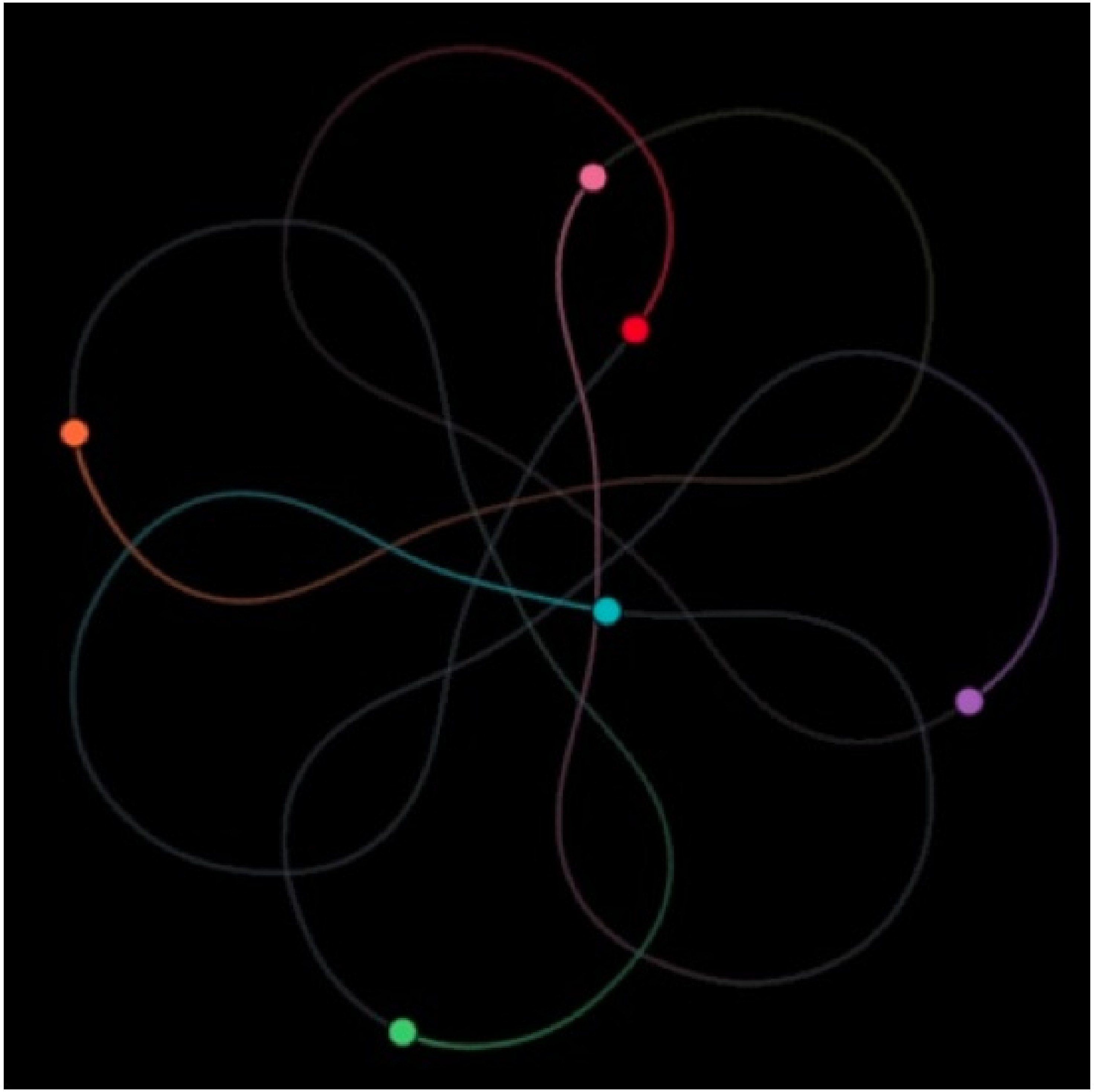
Now, at time $t = 1$ il the third body is at the origin while the first and second are opposite to the origin (collinear configuration).

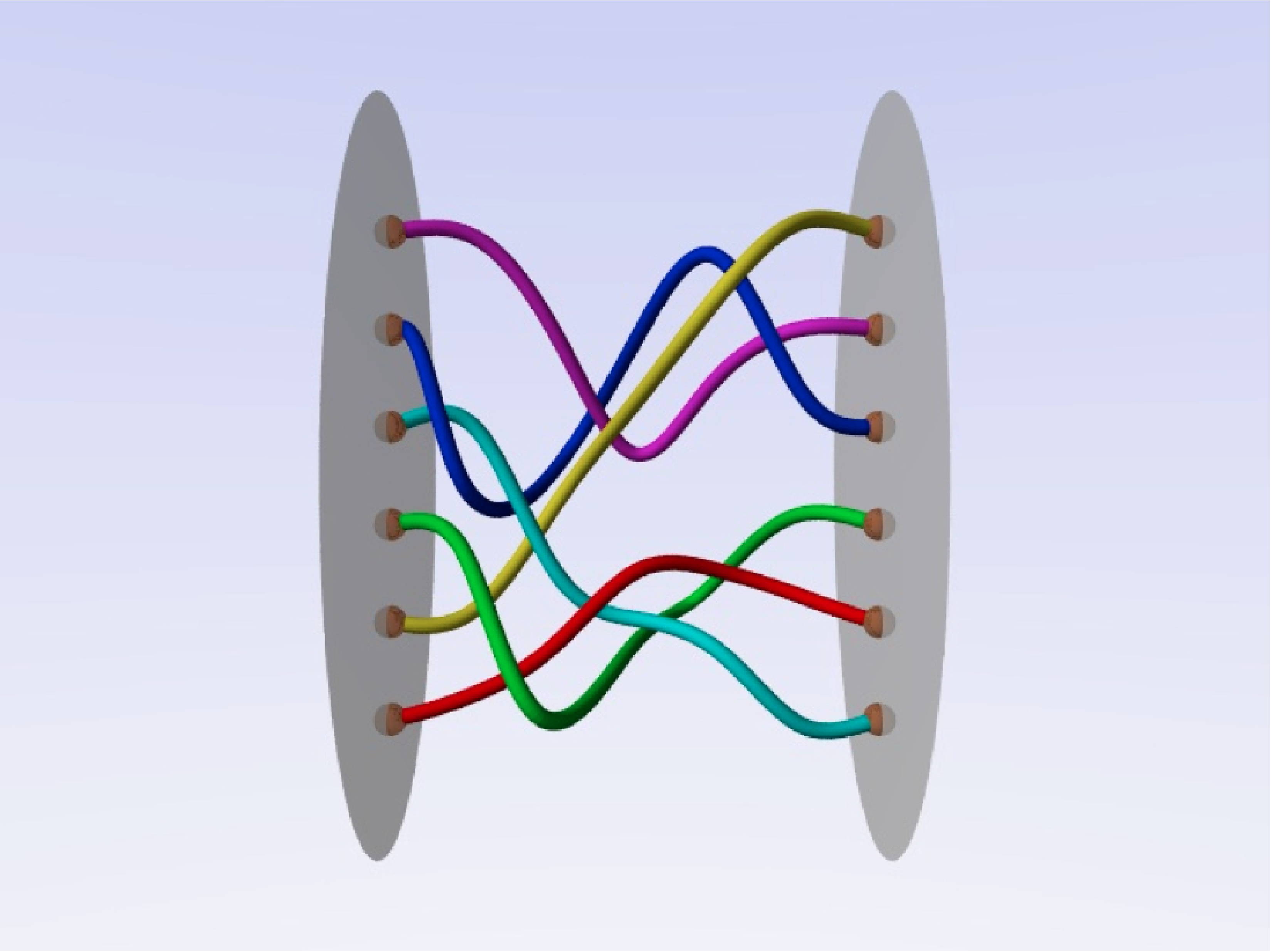
12 The minimal action principle appled to the periodic N–body problem

- Settings: n point particles with masses $m_1, m_2, \dots m_n$ and positions $x_1, x_2, \dots, x_n \in \mathbb{R}^d$, with $d \geq 2$.
- Homogeneous (Newton) potential of degree $-\alpha < 0$ on the configuration space $\mathcal{X}$: $U(x) \simeq \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$.
- On collisions ($x_i = x_j$ for some $i \neq j$), the potential $U = +\infty$.
- T-periodic orbits: solutions of the Newton equations (such that $\forall t : x(t+T) = x(t) \in \mathcal{X}$).
$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}.$$
- Lagrangian: $L(x, \dot{x}) = L = K + U = \sum_i \frac{1}{2} m_i |\dot{x}_i|^2 + \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$.
- Action functional: $\mathcal{A}(x) = \int_0^T L(x(t), \dot{x}(t)) dt$.
- Minimize $\mathcal{A}(x)$ on the class of paths joining any pair of symmetric configurations.

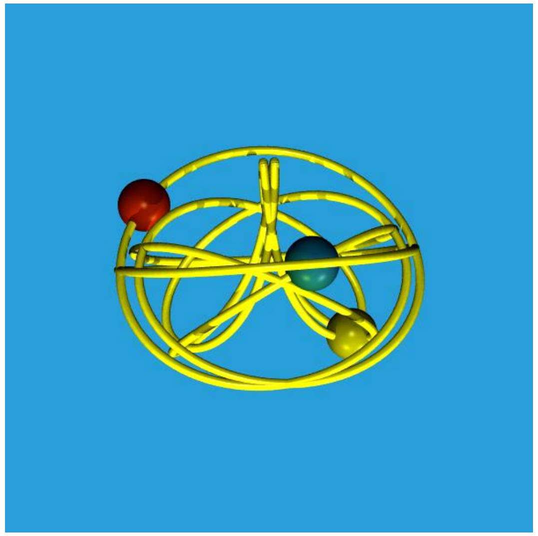
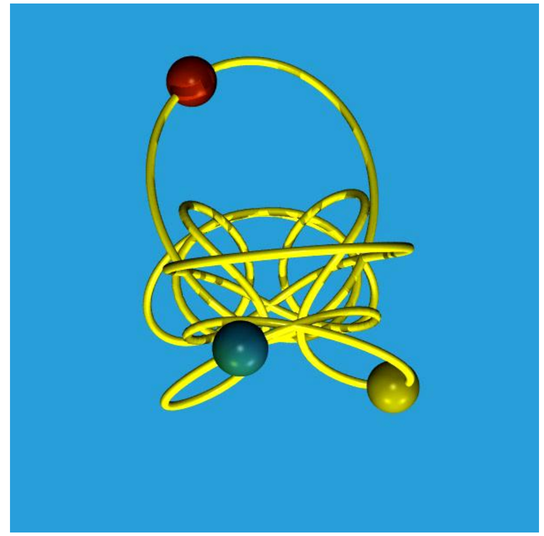
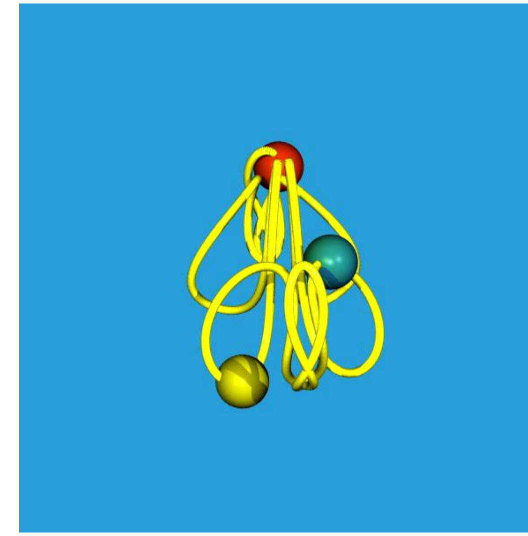


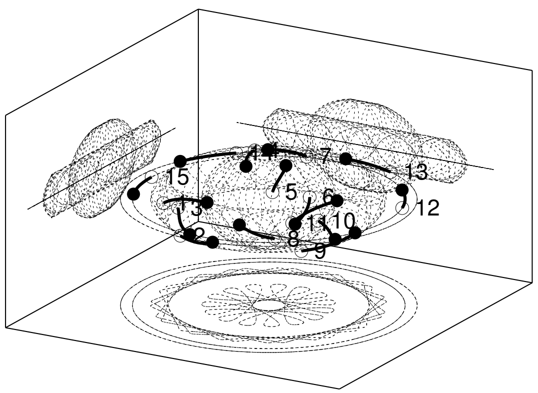
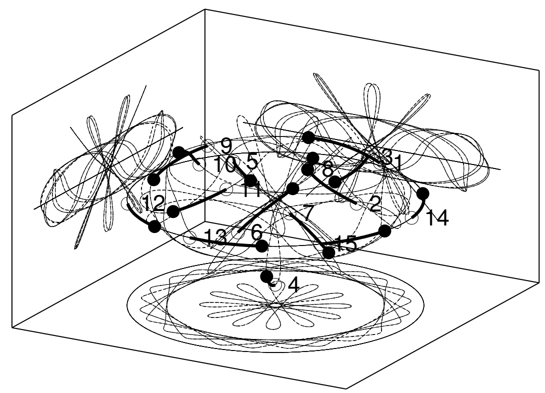
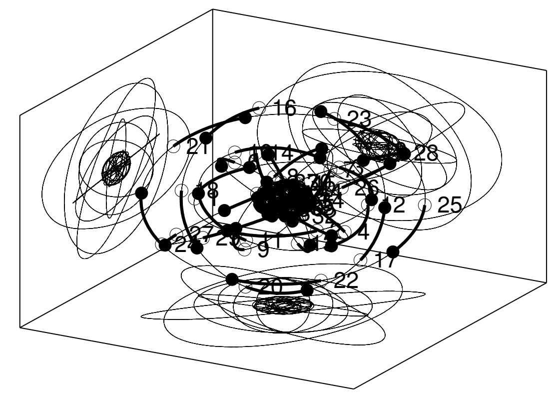
13 Coreographies

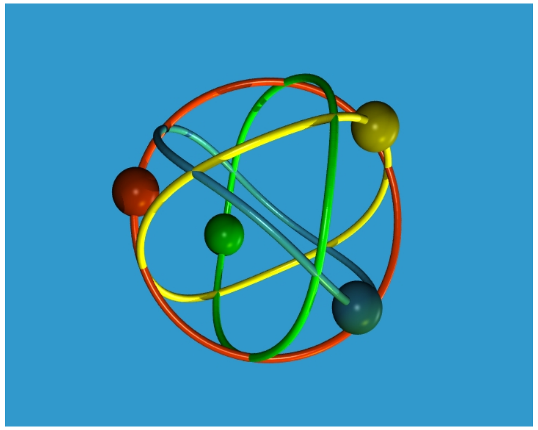


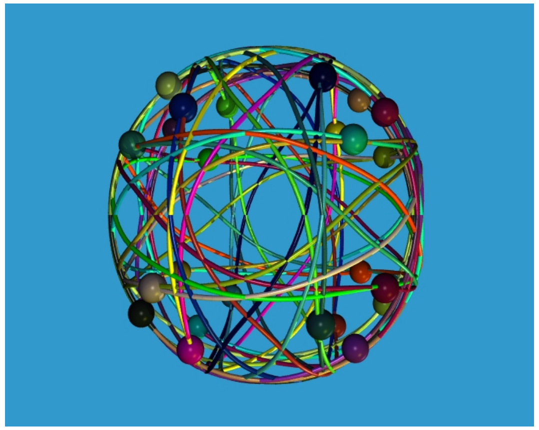


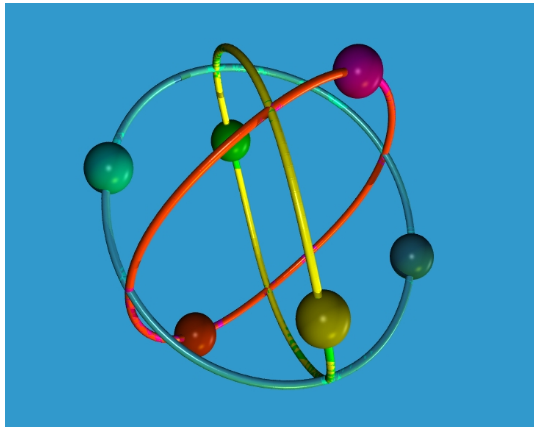
15 Spatial orbits

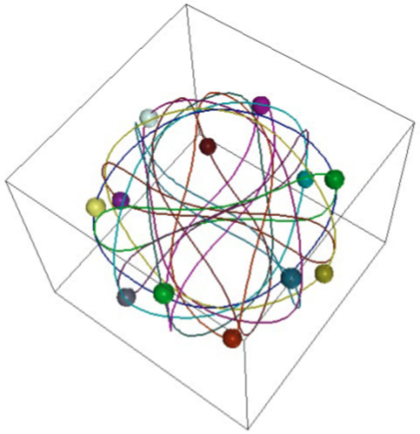




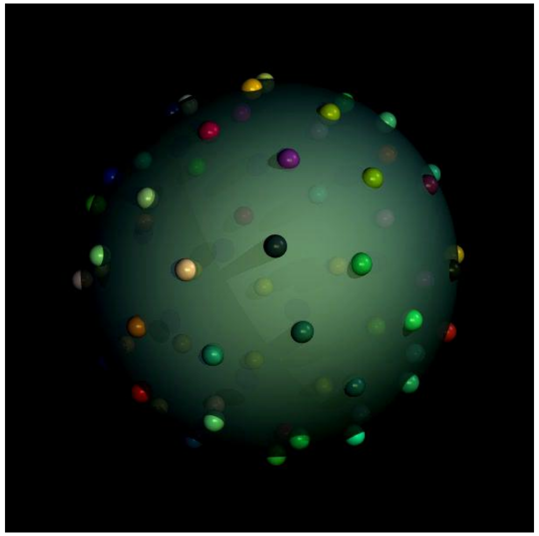





Tetrahedron

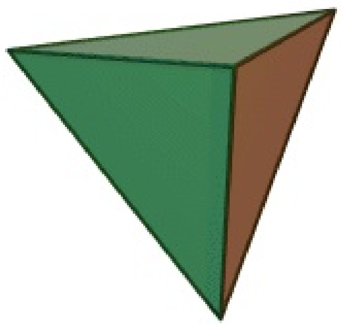

Cube


Octahedron

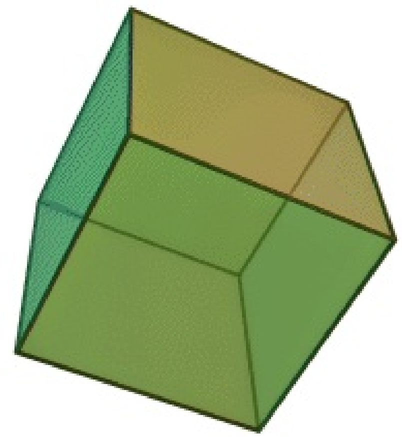

Prism


Dodecahedron

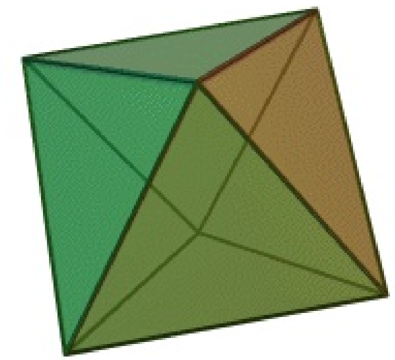
17 Platonic solids



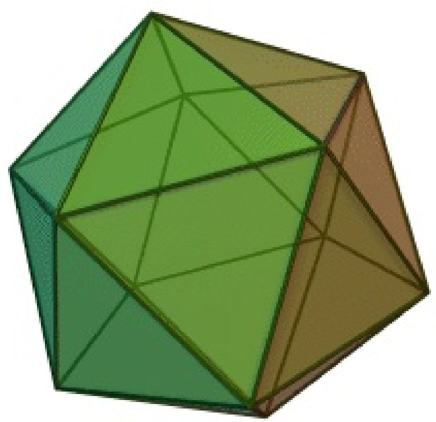
Tetrahedron



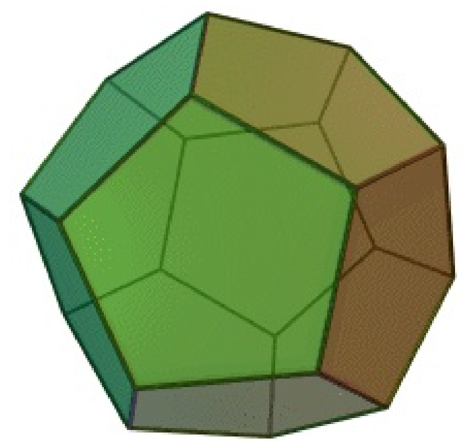
Cube



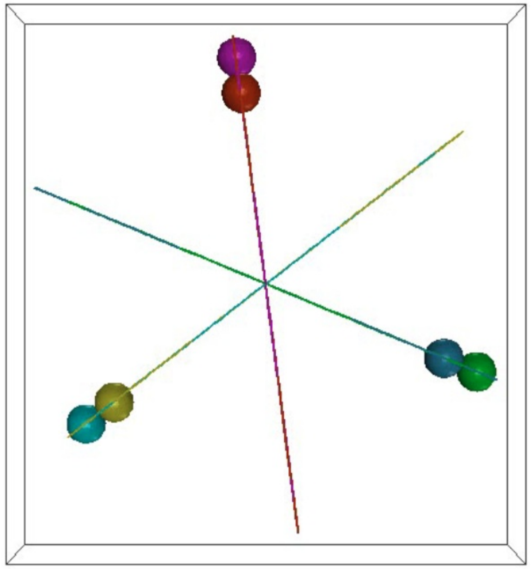
Octahedron

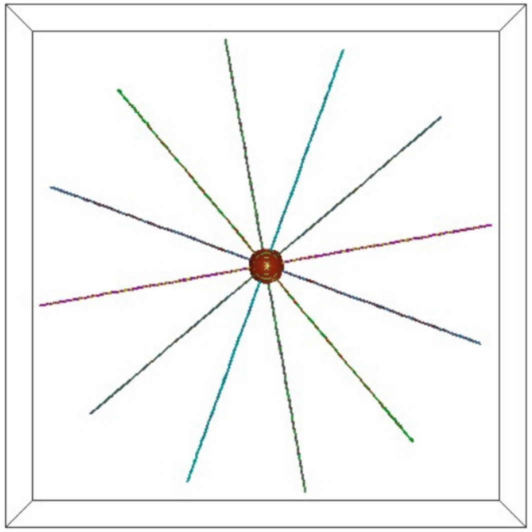


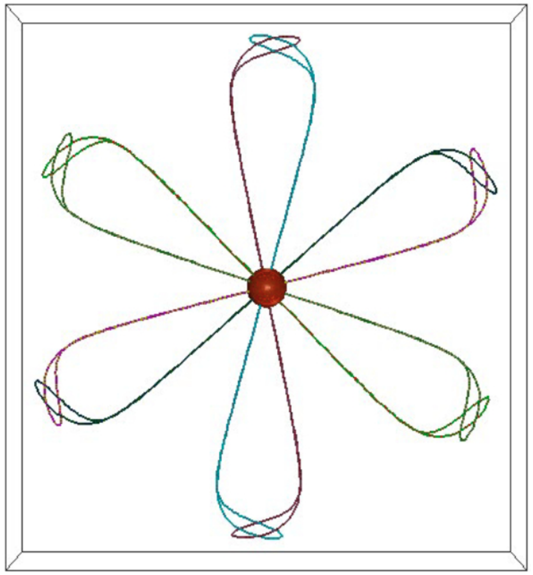
Icosahedron

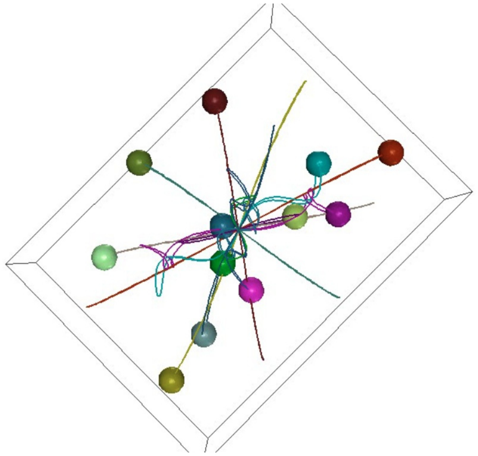


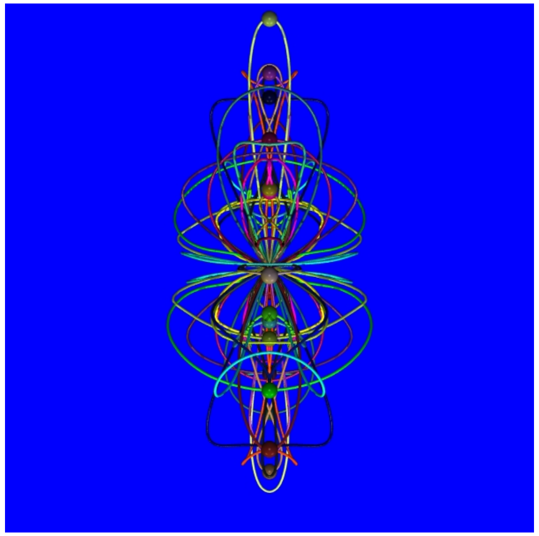
Dodecahedron











19 Papers

[1] V. Barutello, S. Terracini and G. Verzini, *Entire Parabolic Trajectories as Minimal Phase Transitions*, Calc. Var. PDE, 49 (2014), no. 1-2, 391-429, (arXiv:1105.3358)

[2] V. Barutello, S. Terracini and G. Verzini, *Entire Minimal Parabolic Trajectories: the planar anisotropic Kepler problem*, Arch. Rat. Mech. Anal., 207, n. 2 (2013), 583–609, DOI: 10.1007/s00205-012-0565-9 (arXiv:1109.5504)

[3] N. Soave and S. Terracini, *Symbolic Dynamics for the N-centre problem at negative energies*, DCDS-A 32 (2012), 3201–3345

[4] V. Barutello, Ferrario D.L., Terracini S., *On the singularities of generalized solutions to n–body type problem*, Int Math Res Notices.2008 (2008), rnn069-78

[5] V. Barutello, Ferrario D.L., Terracini S., *Symmetry groups of the planar three-body problems and action-minimizing trajectories*, Arch. Rational Mech. Anal. 190 (2008), 189-226

[6] Barutello V, Terracini S., *Double choreographical solutions for n-body type problems*, Celestial Mechanics and Dynamical Astronomy , 95 (2006), 1-4, 67–80

[7] Barutello V., Terracini S., *Action Minimizing Orbits in the N-body problem with simple choreography contstraint*, Nonlinearity 17 (2004), 2015-2039

[8] Ferrario, Davide L.; Terracini, Susanna, *On the Existence of Collisionless Equivariant Minimizers for the Classical n-body Problem*, Invent. Math. 155 (2004), no. 2, 305–362

Related papers adopting a variational approach by: Ambrosetti, Bahri, Barutello, Bessi, Chenciner, Chen, Coti Zelati, Degiovanni, Desolneux, Giannoni, Gordon, Majer, Marchal, Marino, Montgomery, Rabinowitz, Riahi, Sbano, Serra, Tanaka, Terracini, Venturelli.

Singular systems

We consider systems of interacting bodies of the form

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}(t, x), \qquad i = 1, \ldots, n$$

where the forces $\frac{\partial U}{\partial x_i}$ are undefined on a singular set Δ.

➡ Example: the set of collisions between two or more particles in the n–body problem:

$$U(x) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$$

$$\Delta = \bigcup_{i \neq j} \{(x_1, \ldots, x_n) \ : \ x_i = x_j\}$$

Such singularities play a fundamental role in the phase portrait and strongly influence the global orbit structure, as they can be held responsible, among others, of the presence of chaotic motions and of motions becoming unbounded in a finite time (Diacu, Devaney, Gerver, Gutzwiller, Mather, Saari, Simò, Xia). Morevore, singularities are intimately linked to the variational structure of periodic trajectories and to their action spectrum.

21 Key points in the analysis of the impact of the bounded singularities in the n–body problem

- the asymptotic analysis along a single collision trajectory (total or partial); this analysis goes back, in the classical case, to the works by Sundman, Wintner and, in more recent years by Diacu, Sperling, Pollard, Saari, Diacu and other authors;
- blowing–up the singularity by a suitable change of coordinates (named after McGehee) and replacing it by an invariant boundary –the collision manifold– where the flow can be extended in a smooth manner;
- it turns out that, in many interesting applications, the flow on the collision manifold has a simple structure: it is a gradient–like, Morse–Smale flow featuring a few stationary points and heteroclinic connections;
- the analysis of the extended flow allows us to obtain a full picture of the behavior of solutions near the singularity, despite the flow fails to be fully regularizable (except in a few cases).

The variational approach to the periodic N–body problem

- Settings: n point particles with masses $m_1, m_2, \dots m_n$ and positions $x_1, x_2, \dots, x_n \in \mathbb{R}^d$, with $d \geq 2$.
- Homogeneous (Newton) potential of degree $-\alpha < 0$ on the configuration space $\mathcal{X}$: $U(x) \simeq \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$.
- Many results can be extended to logarithmic potentials:almost parallel N–vortex problem;
- On collisions ($x_i = x_j$ for some $i \neq j$) potential $U = +\infty$.
- T-periodic orbits: solutions of the Newton equations (such that $\forall t : x(t+T) = x(t) \in \mathcal{X}$).

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}.$$

- Lagrangian: $L(x, \dot{x}) = L = K + U = \sum_i \frac{1}{2} m_i |\dot{x}_i|^2 + \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$.
- Action functional: $\mathcal{A}(x) = \int_0^T L(x(t), \dot{x}(t)) dt$.

23 Critical points of the action functional

- Sobolev space of T-periodic trajectories: $\Lambda = H^1(\mathbb{T}, \mathcal{X})$.
- Find critical points of action functional

$$\mathcal{A}\colon \Lambda \to \mathbb{R} \cup \infty,$$

constrained on suitable linear subspaces $\Lambda_0 \subset \Lambda$ (natural constraints for $\mathcal{A}$).

Main problems:

- The action functional $\mathcal{A}$ is not coercive on Λ. The minimum needs not to be achieved.
- We can seek critical point others than minimizers: e.g.
  - Local minimizers
  - Constrained minimizers
  - Other type of critical points (mountain pass).
- The action functional $\mathcal{A}$ does not satisfy the Palais Smale condition on Λ: sequences of almost–critical points may diverge.
- The potential U is singular on collisions, and thus miminizers or other critical points can *a priori* be collision trajectories.

Symmetry groups and equivariant orbits

- G finite group.
- $\tau\colon G \to O(2)$ orthogonal representation of dimension 2 (on cyclic time $\mathbb{T} = \mathbb{R} \mod T \cong S^1$).
- $\rho\colon G \to O(d)$ orthogonal representation (on the euclidean space $\mathbb{R}^d$).
- $\sigma\colon G \to \Sigma_n$ homomorphism on the symmetric group on n elements ($\implies$ G on the index set $\mathbf{n} = \{1, 2, \ldots, n\}$)
-
  - G acts on time (translation and reversal) $\mathbb{T}$ via τ;
  - G acts on the configuration space $\mathcal{X}$ via ρ and σ:

$$\forall i = 1 \ldots n : (gx)_i = \rho(g) x_{\sigma(g)^{-1}(i)}.$$

- Consider the linear subspace $\Lambda_0 = \Lambda^G \subset \Lambda$ of periodic trajectories in Λ which are equivariant with respect to the G-action:

$$\forall g \in G : x(gt) = (gx)(t).$$

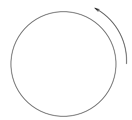
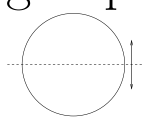
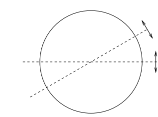
- Consequences:

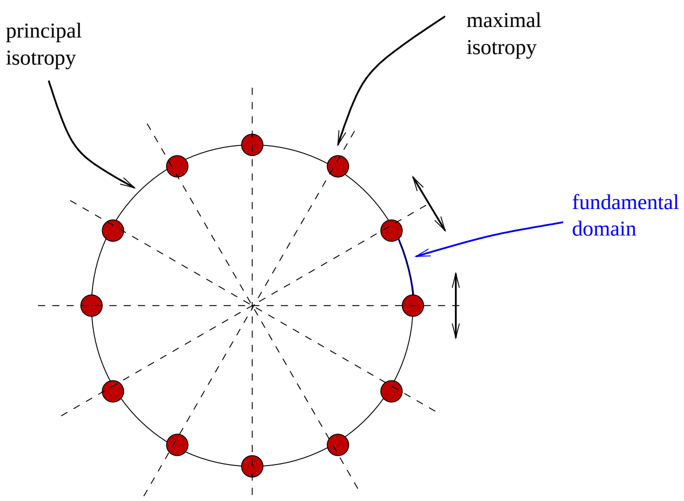
  (1) Λ^G is a natural constraint if $m_{\sigma(g(i))} = m_i$, for all i and $g \in G$.

  (2) Gain of coercivity: if $\mathcal{X}^G$ is trivial we have $(x \in \Lambda_0, |x| \to \infty \implies \mathcal{A}(x) \to \infty)$.

25 Cyclic and dihedral actions

Consider the normal subgroup $\ker\tau \lhd G$ and the quotient $\bar{G} = G/\ker\tau$. Since $\bar{G}$ acts effectively on $\mathbb{T}$, it is either a *cyclic* group or a *dihedral* group.

- If the group $\bar{G}$ acts trivially on the orientation of $\mathbb{T}$, then $\bar{G}$ is cyclic and we say that the action of G on Λ is of cyclic type. 
- If the group $\bar{G}$ consists of a single reflection on $\mathbb{T}$, then we say that action of G on Λ is of brake type. 
- Otherwise, we say that the action of G on Λ is of dihedral type. 



If $\mathbb{I} = [0, 1]$ is the fundamental domain (for a dihedral type), then G-equivariant trajectories correspond to paths $x\colon \mathbb{I} \to \mathcal{X}^{\ker\tau}$ with $x(0) \in \mathcal{X}^{H_0}$ and $x(1) \in \mathcal{X}^{H_1}$, where H_0 and H_1 are the maximal isotropy subgroups of the boundary of $\mathbb{I}$.

- G-equivariance can be split into:
  - proper boundary conditions on the fundamental domain;
  - a time-independent constraint on the space of configurations.

We can prove the existence of a multiplicity of periodic trajectories by a systematic use of equivariant variational methods. This involves:

- The classification of all the admissible symmetry groups.
- The analysis of possible collisions for equivariant minimizers and the determination of those groups whose minimizers are free of collisions.
- A further study of qualitative properties of equivariant minimizers to understand whether different classes of symmetric loops may share the same minimizers.

Further possible developments

(1) Develop an equivariant Morse Theory specific for the N–body problem, taking into account of all possible collisions.

(2) Fully understand the impact of collisions on the variational characterization (Morse index) of periodic trajectories.

27 Absence of collision for locally minimal paths

As a matter of fact, solutions to the Newtonian n–body problem which are minimals for the action are, very likely, free of any collision. This fact was observed by the construction of suitable local variation arguments for the 2 and 3–body cases by Serra and Terracini (1992 and 1994). The 4–body case was treated afterward by Dell'Antonio (non really rigorously) and then by A. Venturelli in his PhD thesis. In general, the proof goes by the sake of the contradiction and involves the construction of a suitable variation that lowers the action in presence of a collision. A recent breakthrough in this direction is due of the neat idea, due to C. Marchal, of averaging over a family of variations parameterized on a sphere. The method of averaged variations for Newtonian potentials has been developed and exposed by Chenciner, and then extended to α–homogeneous potentials and various constrained minimization problems by Ferrario and Terracini. This argument can be used in many of the known cases to prove that minimizing trajectories are collisionless.

Problems:

- Anisotropic and logarithmic potentials.
- Study the contributions to the Morse index given by the possible collisions (Barutello, Secchi, (2006).

Let G_0 be the isotropy group at the collision time, then the blow–up procedure implies the existence of q, a G_0-equivariant minimizing homothetic collision trajectory.

The *standard variation* associated to δ and T is defined as

$$v^\delta(t) = \begin{cases} \delta & \text{if } 0 \leq |t| \leq T - |\delta| \\ (T-t)\frac{\delta}{|\delta|} & \text{if } T - |\delta| \leq |t| \leq T \\ 0 & \text{if } |t| \geq T. \end{cases}$$

Our next goal is to find a G_0-equivariant standard variation v^δ such that the trajectory $q + v^\delta$ does not have a collision at $t = 0$ and

$$\boxed{\Delta\mathcal{A} := \int_{-\infty}^{+\infty} [\mathcal{L}_{\mathbf{k}}(q + v^\delta) - \mathcal{L}_{\mathbf{k}}(q)]dt < 0.}$$

Introduce the potential displacement function

$$\boxed{S(\xi, \delta) = \int_0^{+\infty} \left(\frac{1}{\left|\xi t^{2/(2+\alpha)} - \delta\right|^\alpha} - \frac{1}{\left|\xi t^{2/(2+\alpha)}\right|^\alpha} \right) dt}$$

where $\xi, \delta \in \mathbb{R}^2$.

Theorem: *Let $q = \{q\}_i = \{t^{2/(2+\alpha)}\xi_i\}$, $i = 1, \dots, k$ be a parabolic collision trajectory and v^δ a G_0-equivariant standard variation. Then, as $\delta \to 0$*

$$\Delta\mathcal{A} = 2|\delta|^{1-\alpha/2} \sum_{\substack{i<j \\ i,j\in\mathbf{k}}} m_i m_j S(\xi_i - \xi_j, \frac{\delta_i - \delta_j}{|\delta|}) + O(|\delta|).$$

We observe that

$$S(\lambda\xi, \mu\delta) = |\lambda|^{-1-\alpha/2} |\mu|^{1-\alpha/2} S(\xi, \delta)$$

and hence the sign of S depends on the angle between ξ and δ. Let

$$\Phi(\vartheta) = \int_0^{+\infty} \frac{1}{\left(t^{\frac{4}{\alpha+2}} - 2\cos\vartheta t^{\frac{2}{\alpha+2}} + 1\right)^{\alpha/2}} - \frac{1}{t^{\frac{2\alpha}{\alpha+2}}} dt, \quad \alpha \in (0, 2)$$

$\Phi(\theta)$ represents the potential differential needed for displacing the colliding particle from zero to $e^{i\theta}$. Expanding, we find

$$\Phi(\vartheta) = \frac{\alpha(\alpha+2)}{2} \left\{ \frac{1}{\alpha-2} \beta\left(\frac{\alpha+2}{4}, \frac{\alpha+2}{4}\right) + \frac{1}{\alpha} \sum_{k=1}^{+\infty} \binom{-\alpha/2}{k} (-1)^k 2^{k-1} (\cos\vartheta)^k \beta\left(\frac{\alpha}{4} - \frac{1}{2} + \frac{k}{2}, \frac{\alpha}{4} + \frac{1}{2} + \frac{k}{2}\right) \right\}.$$

29 Some properties of Φ

The value of $\Phi(\theta)$ ranges from $+\infty$ to some negative value, depending on α. However, thanks to some harmonic analysis one can prove that suitable *averages* are always negative: the first inequality is particularity useful for dealing with reflected triple collisions from the Lagrange central configuration:

$$\Phi(\frac{2\pi}{3}+\gamma)+\Phi(\frac{2\pi}{3}-\gamma)<0, \quad \forall\gamma\in[0,\pi/2].$$

A key remark was made by Christian Marchal: being the Newton potential a harmonic map averaging it on a sphere results in a truncation in the interior. In fact, is not so much a matter of harmonicity. A crucial estimate was proved in [FT] about the averages of Φ on circles:

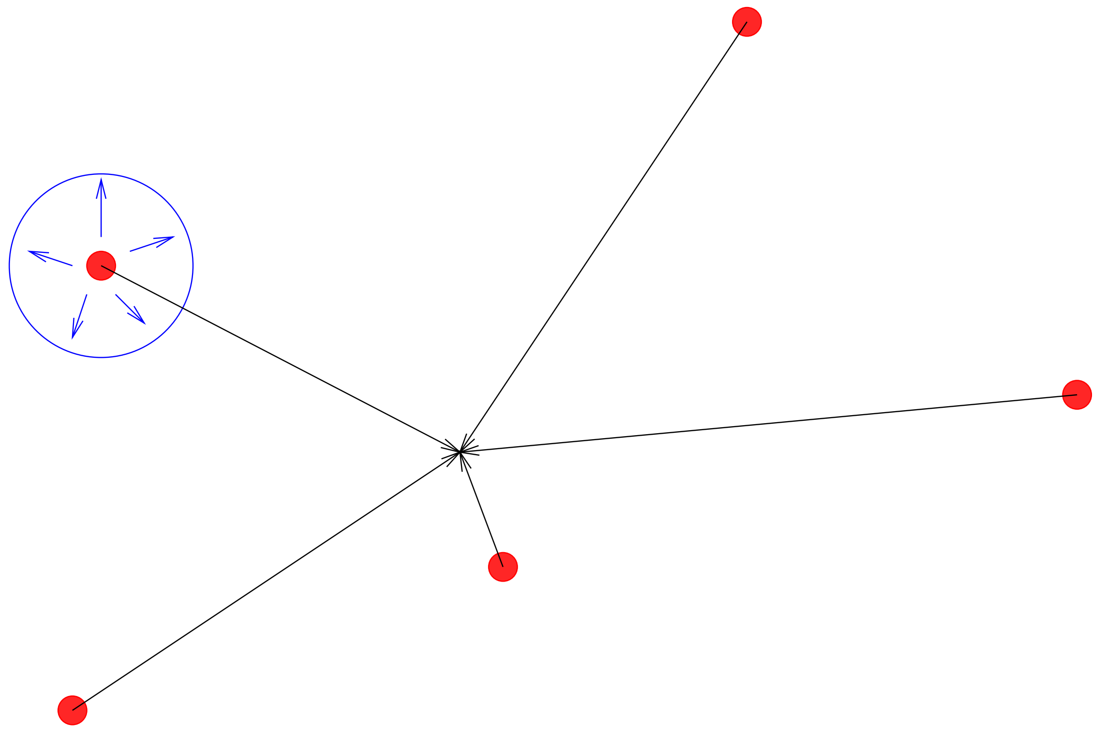
For every $\alpha>0$, $\xi\in\mathbb{R}^3\setminus\{0\}$ and for every circle $\mathbb{S}\subset\mathbb{R}^d$ with center in 0,

$$\tilde{S}(\xi,\mathbb{S})=\frac{1}{|\mathbb{S}|}\int_{\mathbb{S}}S(\xi,\delta)d\delta=|\xi|^{-1-\alpha/2}|\delta|^{1-\alpha/2}\frac{1}{2\pi}\int_0^{2\pi}\Phi(\theta)d\theta<0.$$

Consider $\xi=x_i-x_j$ and δ ranging in a circle. Then we obtain the principle, a generalization of the result announced in :

Chenciner, A., Action minimizing solutions of the Newtonian n–body problem: from homology to symmetry, August 2002, *ICM, Peking*

Marchal's Principle



It is more convenient (from the point of view of the integral of the potential on the time line) to replace one of the point particles with a homogeneous circle of same mass and fixed radius which is moving keeping its center in the position of the original particle

If the action of G on $\mathbb{T}$ and $\mathcal{X}$ fulfills some conditions (computable) then (local) minimizers of the action functional $\mathcal{A}^G$ in $\Lambda^G \subset \Lambda$ do not have collisions.

31 The function $\tilde{S}(\xi, \mathbb{S})$ as a hypergeometric combination

We can write $\tilde{S}(\xi, \mathbb{S})$ in terms of hypergeometric functions as follows:

$$\tilde{S}(\xi, \mathbb{S}) = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 + \int_1^{+\infty} \right) \left[\frac{1}{|\xi t^{2/(2+\alpha)} + \delta|^\alpha} - \frac{1}{|t^{2/(2+\alpha)}\xi|^\alpha} \right] dt\, d\theta$$

We have

$$\tilde{S}(\xi, \mathbb{S}) = {}_3F_2 \left(\begin{matrix} \alpha/2, & \alpha/2, & (2+\alpha)/4; \\ & 1, & (6+\alpha)/4; \end{matrix} \, 1 \right) - \frac{2+\alpha}{2-\alpha} +$$
$$\frac{2+\alpha}{2-\alpha} \left(1 - {}_3F_2 \left(\begin{matrix} \alpha/2, & \alpha/2, & (\alpha-2)/4; \\ & 1, & (\alpha+2)/4; \end{matrix} \, 1 \right) \right).$$

They are nearly-poised (of the second kind) hypergeometric functions evaluated in 1. They are balanced (i.e. Saalschützian) if and only if $\alpha = 1$.

$$\tilde{S}(\xi, \mathbb{S}) dt = \frac{2+\alpha}{4} \sum_{k=0}^{\infty} \left[\binom{-\alpha/2}{k}^2 \frac{1}{k + \frac{\alpha+2}{4}} \left(\frac{(\alpha/2+k)^2}{(1+k)^2} + 1 \right) \right] - \frac{2+\alpha}{2-\alpha}.$$

The rotating circle property

For a group H acting orthogonally on $\mathbb{R}^d$, a circle $\mathbb{S} \subset \mathbb{R}^d$ (with center in 0) is termed *rotating under H* if $\mathbb{S}$ is invariant under H (that is, for every $g \in H$ $g\mathbb{S} = \mathbb{S}$) and for every $g \in H$ the restriction $g|\mathbb{S}\colon \mathbb{S} \to \mathbb{S}$ is a rotation (the identity is meant as a rotation of angle 0).

Let $i \in \mathbf{n}$ be an index and $H \subset G$ a subgroup. A circle $\mathbb{S} \subset \mathbb{R}^d = V$ (with center in 0) is called *rotating for i under H* if $\mathbb{S}$ is rotating under H and

$$\mathbb{S} \subset V^{H_i} \subset V = \mathbb{R}^d,$$

where $H_i \subset H$ denotes the isotropy subgroup of the index i in H relative to the action of H on the index set $\mathbf{n}$ induced by restriction (that is, the isotropy $H_i = \{g \in H \mid gi = i\}$).

A group G acts with the *rotating circle property* if for every $\mathbb{T}$-isotropy subgroup $G_t \subset G$ and for at least $n-1$ indexes $i \in \mathbf{n}$ there exists in $\mathbb{R}^d$ a rotating circle $\mathbb{S}$ under G_t for i.

- If the action has the rotating circle property, then for every $g \in G$ the linear map $1 - g$ sends the rotating circle into another circle (thus we can use the averaging trick).
- In most of the known examples the property is fulfilled.
- There are several infinite families with the rotating circle property.

Theorems with the RCP

- **Theorem**: Consider a finite group K acting on Λ with the rotating circle property. Then a minimizer of the K-equivariant fixed–ends (Bolza) problem is free of collisions.
- **Corollary**: For every $\alpha > 0$, minimizers of the fixed-ends (Bolza) problem are free of interior collisions.
- **Corollary**: If the action of G on Λ is of cyclic type and $\ker \tau$ has the rotating circle property then any local minimizer of $\mathcal{A}^G$ in Λ^G is collisionless.
- **Corollary**: If the action of G on Λ is of cyclic type and $\ker \tau = 1$ is trivial then any local minimizer of $\mathcal{A}^G$ in Λ^G is collisionless.
-

> **Theorem**: Consider a finite group G acting on Λ so that every maximal $\mathbb{T}$-isotropy subgroup of G either has the rotating circle property or acts trivially on the index set $\mathbf{n}$. Then any local minimizer of $\mathcal{A}^G$ yields a collision-free periodic solution of the Newton equations for the n-body problem in $\mathbb{R}^d$.