# Playing with origami

## If interested don't hesitate to contact me!

Andrea Villa
amorvincomni@gmail.com

28th April 2014

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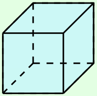

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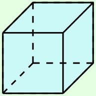

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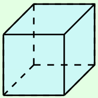

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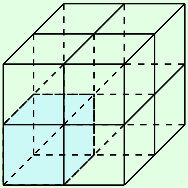

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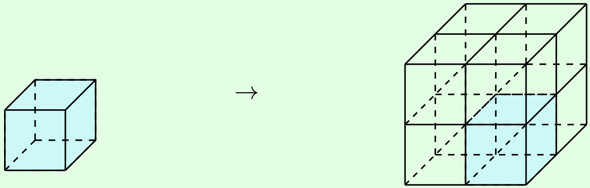

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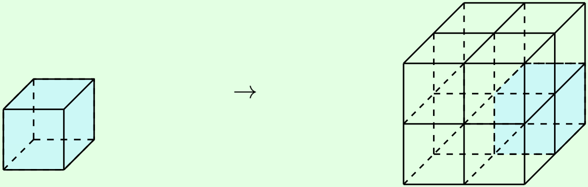

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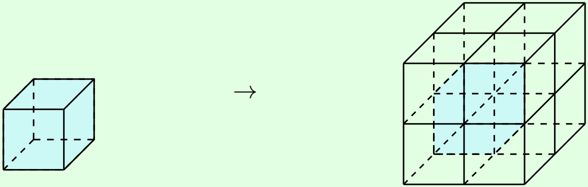

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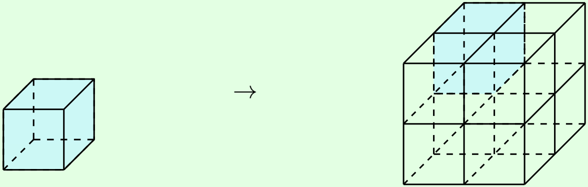

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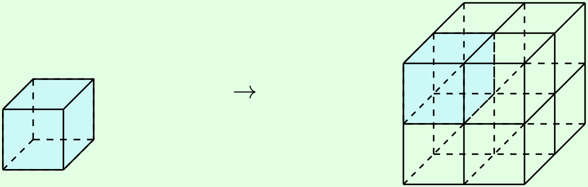

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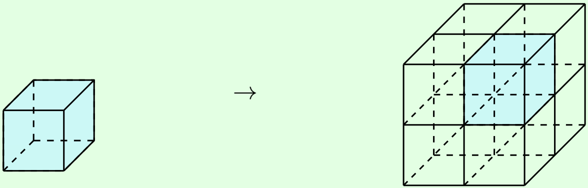

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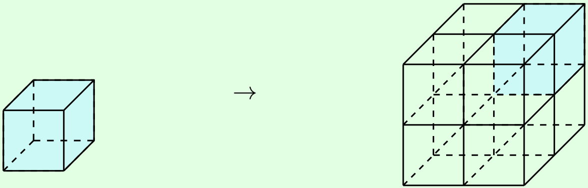

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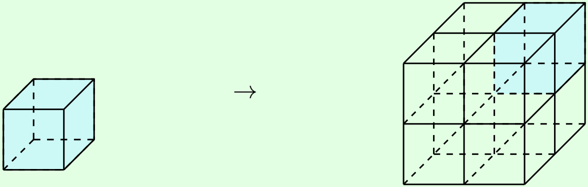
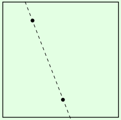
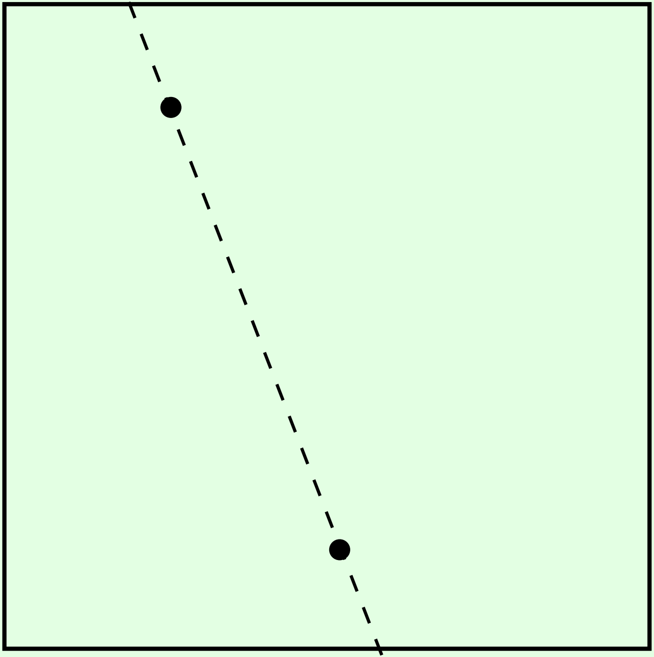


Greek used straight-edge and compass, and they failed

# ORIGAMI RULES:



($L1$) Crease between two different points:



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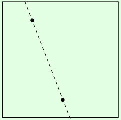

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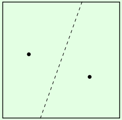
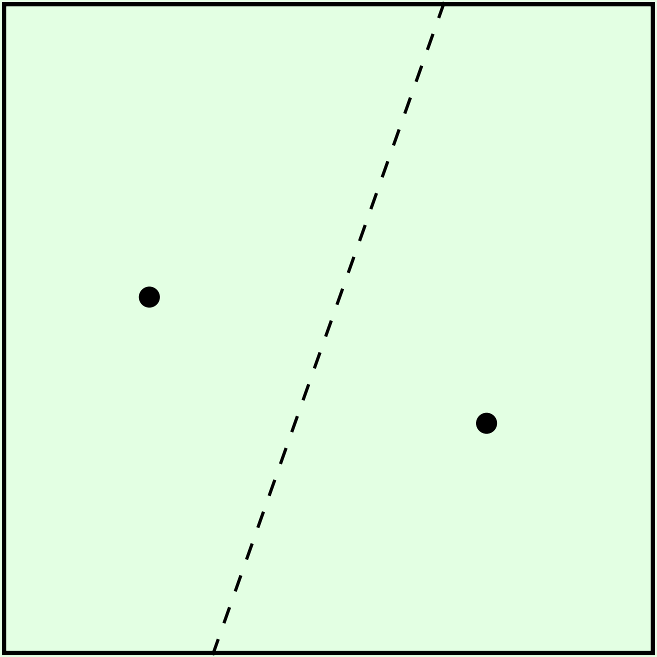


($L2$) Perpendicular bisector of the segment



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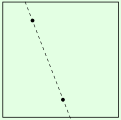

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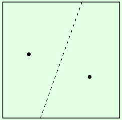

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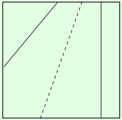
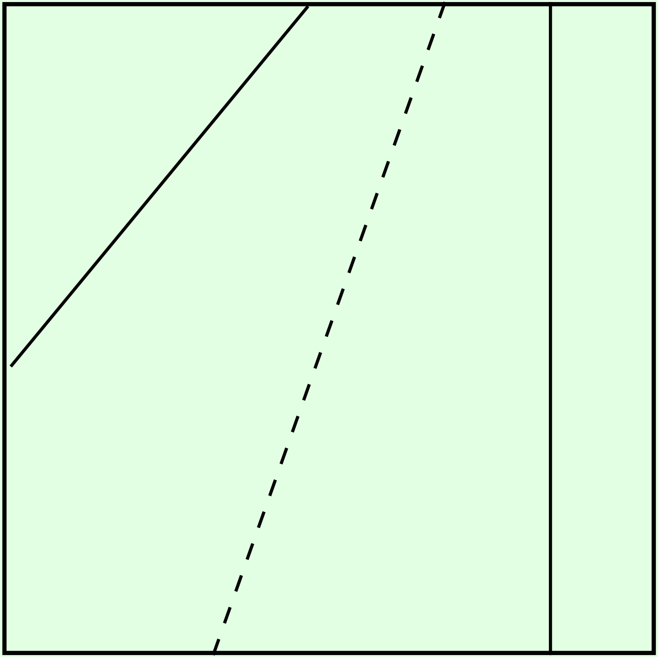


($L3$) Angle bisector of two lines



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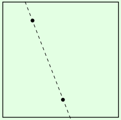

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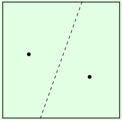

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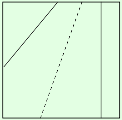
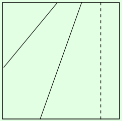
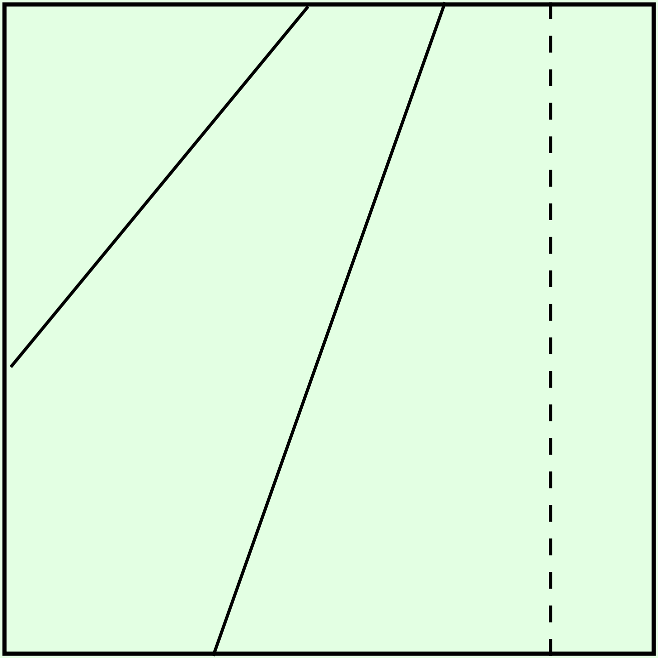


($L3$) Angle bisector of two lines

($L3$) mirror reflection of a line





A line parallel to a given line $r$ through any point $P$.

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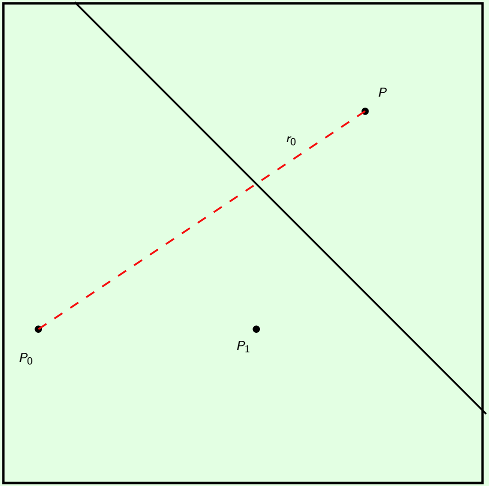


($L1$) fold the crease $r_0$ between $P_0$ and $P$.

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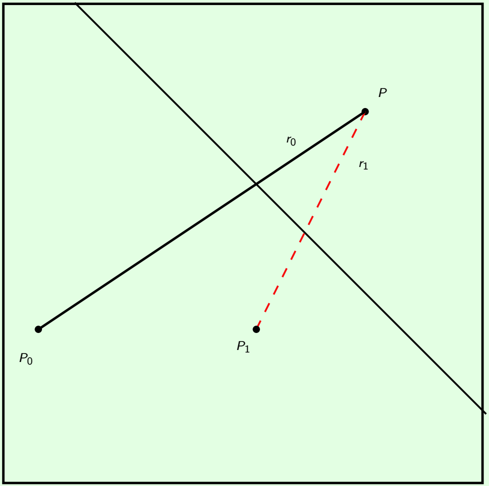


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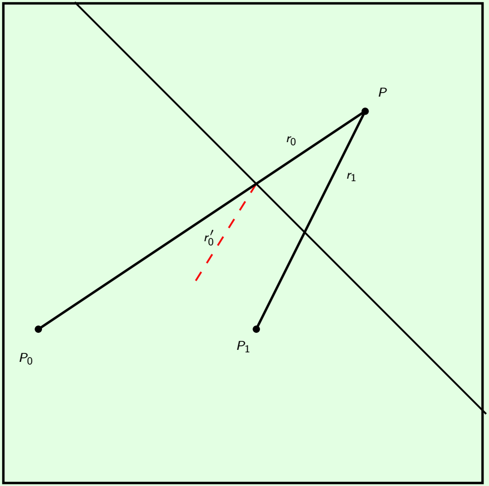
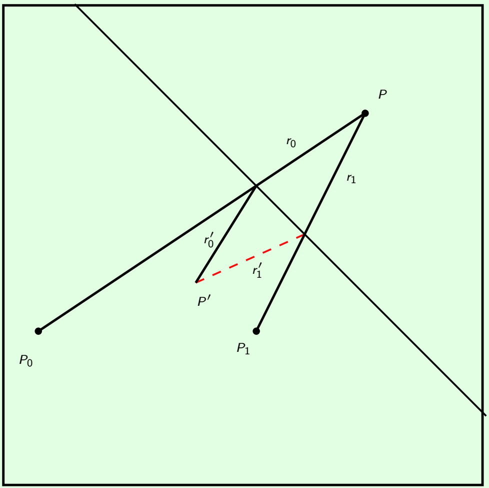



($L4$) reflect the line $r_0$ about line $r$

A line parallel to a given line $r$ through any point $P$.

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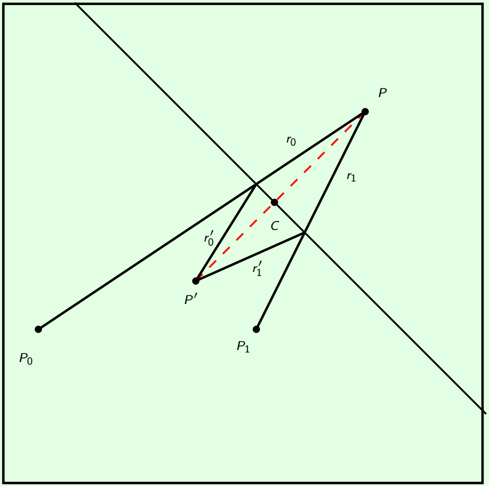
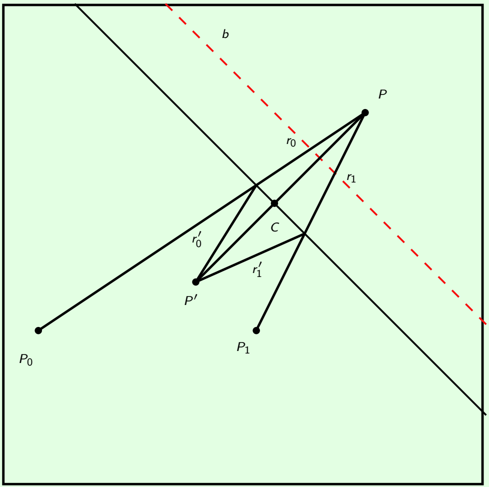


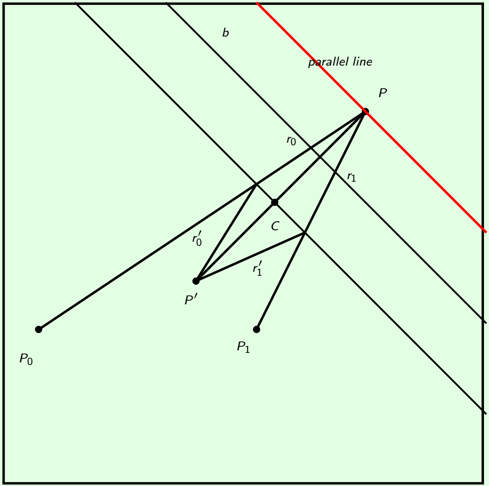

($L1$) fold the crease $h$ between $P'$ and $P$

A line parallel to a given line $r$ through any point $P$.




($L2$) fold the perpendicular bisector of the segment $\overline{CP}$

A line parallel to a given line $r$ through any point $P$.




($L4$) reflect the line $r$ via the new line $b$.

## Definition (Origami Pair)

$(\mathcal{P}, \mathcal{L})$ is an origami pair if $\mathcal{P}$ is a set of points and $\mathcal{L}$ is a set of lines in $\mathbb{R}^2$ satisfying:

(1) Any two non-parallel lines in $\mathcal{L}$ intersect in a point of $\mathcal{P}$.

(2) Any two distinct points in $\mathcal{P}$ the line passing through them is in $\mathcal{L}$.

(3) Any two distinct points in $\mathcal{P}$ the perpendicular bisector of the line segment characterized the points is in $\mathcal{L}$.

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## Definition (Constructible points)

The set of origami constructible points is the smallest subset of $\mathbb{R}^2$ containing the points $(0, 0)$ and $(1, 0)$ that is closed under origami constructions.

## Definition (Origami numbers)

The set of origami numbers is:

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### Definition (Origami numbers)

The set of origami numbers is:
$\mathbb{F}_0 = \{\alpha \in \mathbb{R} \mid \exists v_1, v_2 \in \mathcal{P},\ |\alpha| = \mathrm{dist}(v_1, v_2)\}$.

**Theorem**

*The set of origami numbers is the smallest sub-field of $\mathbb{R}$ closed under operation $x \mapsto \sqrt{1+x^2}$.*

*Proof.*

• inverse • multiplication • $x \mapsto \sqrt{1+x^2}$ $\mathbb{F}_0 \equiv \mathbb{F}_{\sqrt{1+x^2}}$

□

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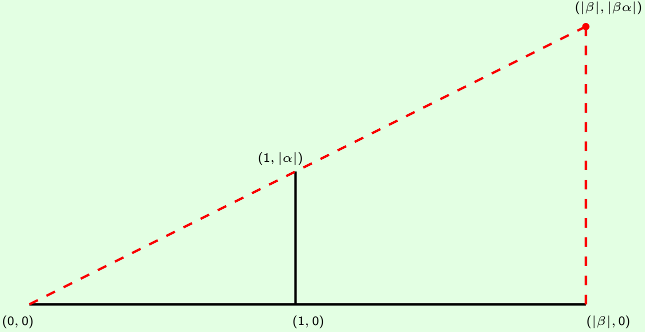


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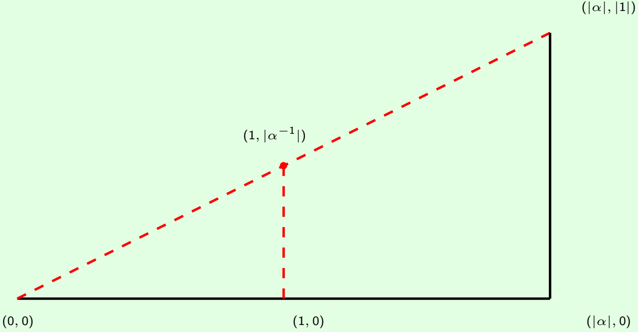


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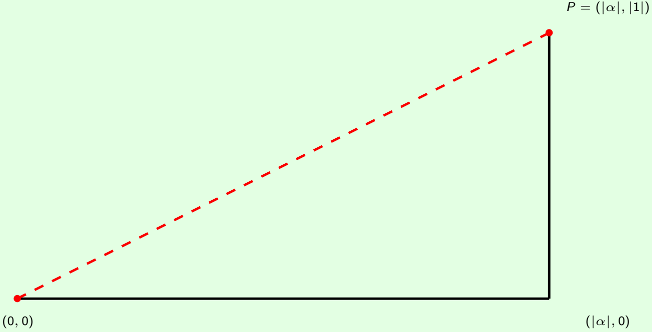
## Theorem

*The set of origami numbers is the smallest sub-field of $\mathbb{R}$ closed under operation $x \mapsto \sqrt{1+x^2}$.*

## Proof.

- inverse
- multiplication
- $x \mapsto \sqrt{1+x^2}$ $\quad \mathbb{F}_0 \equiv \mathbb{F}_{\sqrt{1+x^2}}$

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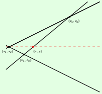


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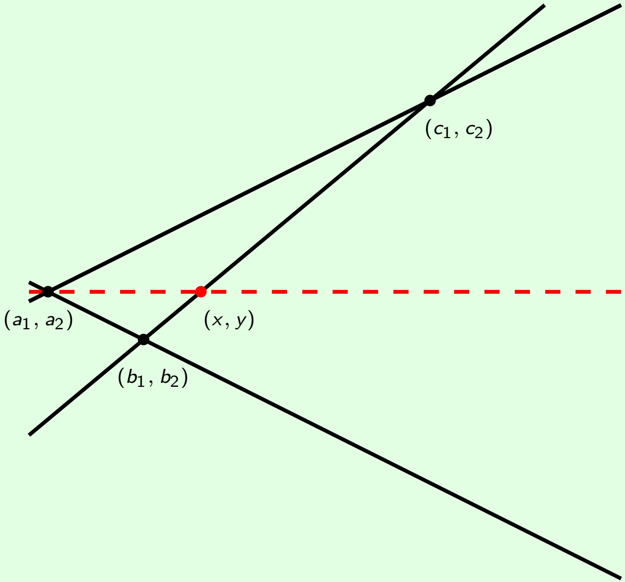
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$(a_1, a_2)$
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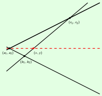


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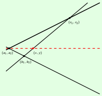


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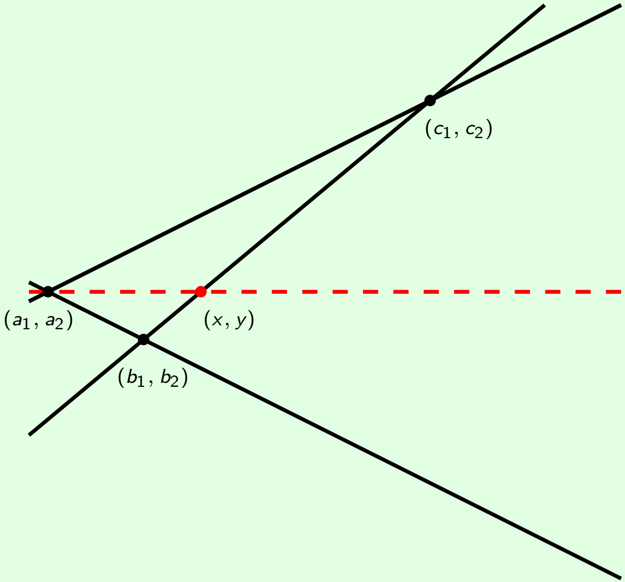
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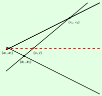


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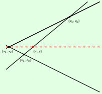


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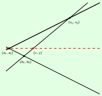


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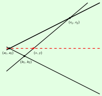


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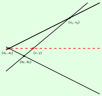


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What about $\sqrt[3]{2}$?

1. A number, $\alpha$, is an algebraic number if it is a root of a polynomial with rational coefficients.
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If $\alpha, \beta \in \mathbb{F}_{TR} \implies -\alpha, \alpha^{-1}, \sqrt{1+\alpha^2}, \alpha+\beta, \alpha \cdot \beta \in \mathbb{F}_{TR}$.

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$$q_{\alpha^{-1}}(x) = \left(\prod_i (x - \alpha_i^{-1})\right)\left(\prod_i \alpha_i\right),$$

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$$q_{\sqrt{1+\alpha^2}}(x) = \prod_i (x^2 - 1 - \alpha_i^2),$$

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$\mathbb{F}_{TR}$ is a field closed under $x \mapsto \sqrt{1+x^2} \iff \mathbb{F}_0 \subset \mathbb{F}_{TR}$

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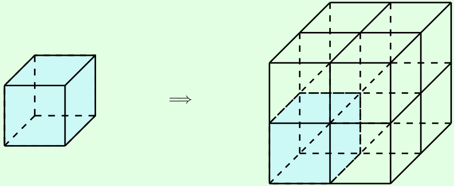

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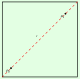
| Line between two points  | |
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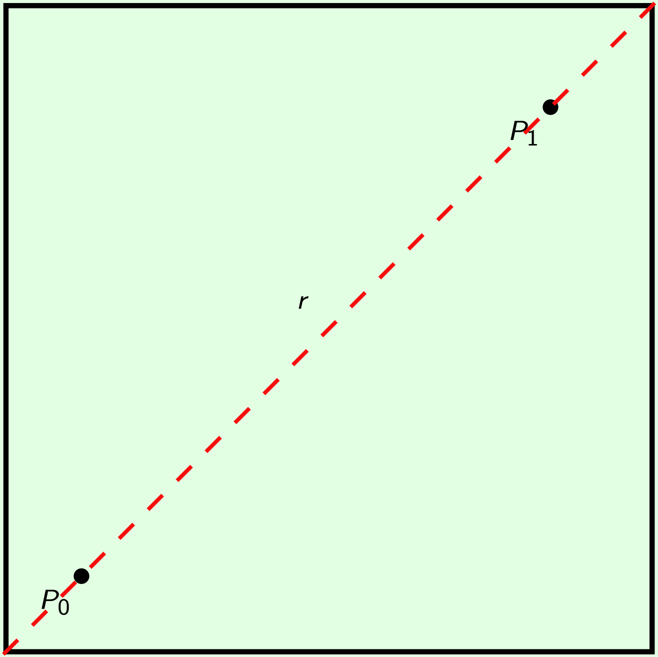
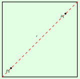
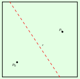

$P_1$
$r$
$P_0$

| Line between two points | Axe of a segment |
|---|---|
|  |  |
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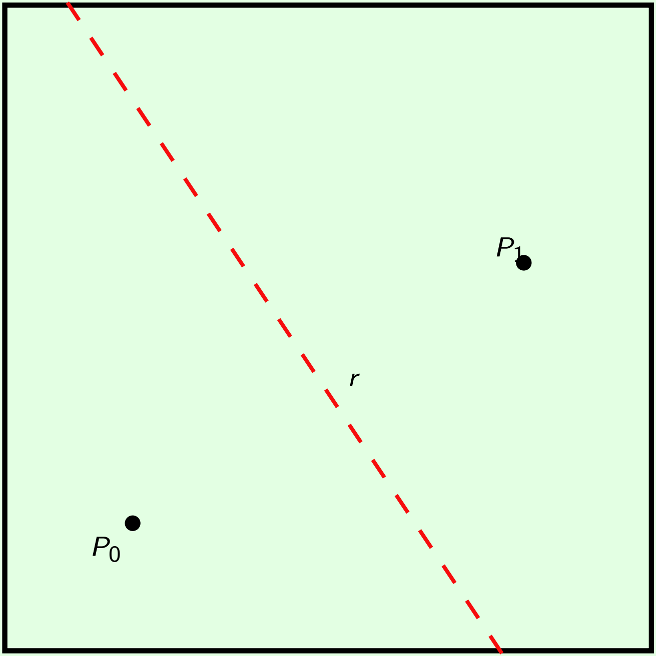
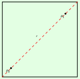
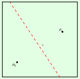
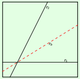


$P_1$
$r$
$P_0$

| | |
|---|---|
| Line between two points  | Axe of a segment  |
| Bisector of an angle  | |
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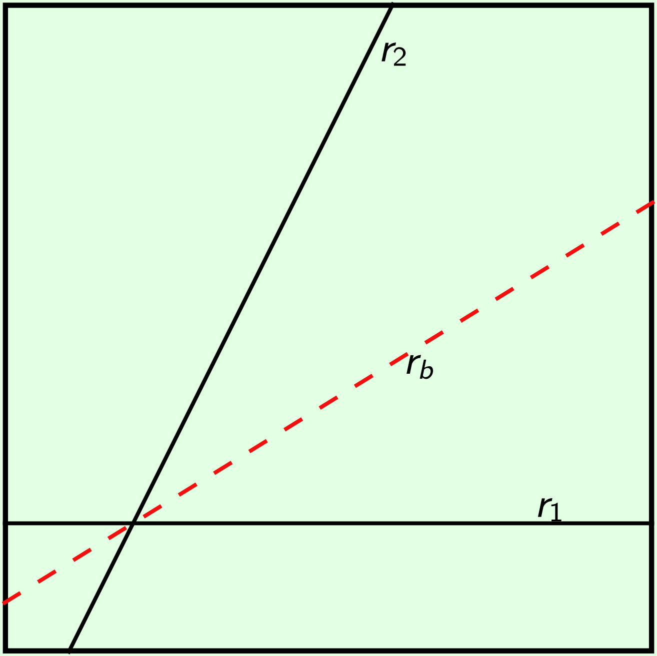
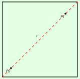
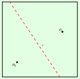
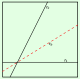
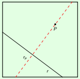


$r_2$
$r_b$
$r_1$

| Line between two points | Axe of a segment |
|---|---|
|  |  |
| Bisector of an angle | Perpendicular through a point |
|  |  |
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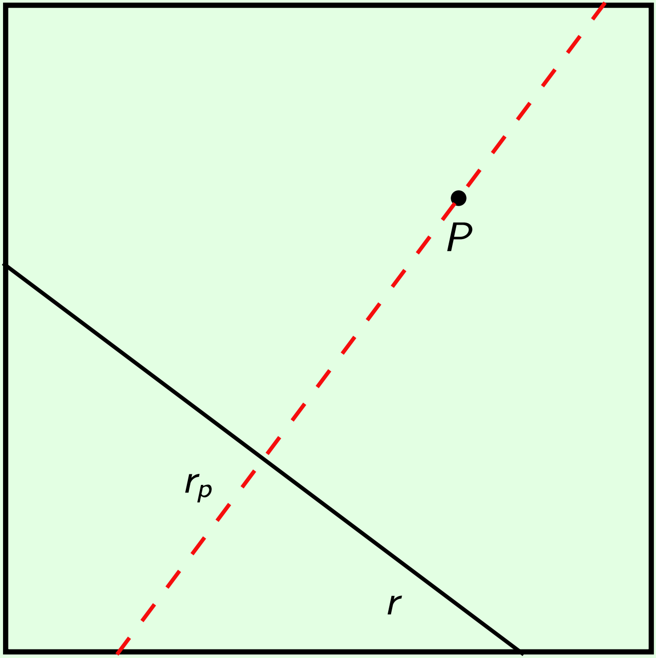
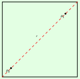
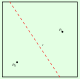
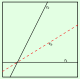
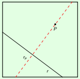
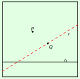


P
$r_p$
$r$

| | |
|---|---|
| Line between two points  | Axe of a segment  |
| Bisector of an angle  | Perpendicular through a point  |
| Tg of a parabola  | |
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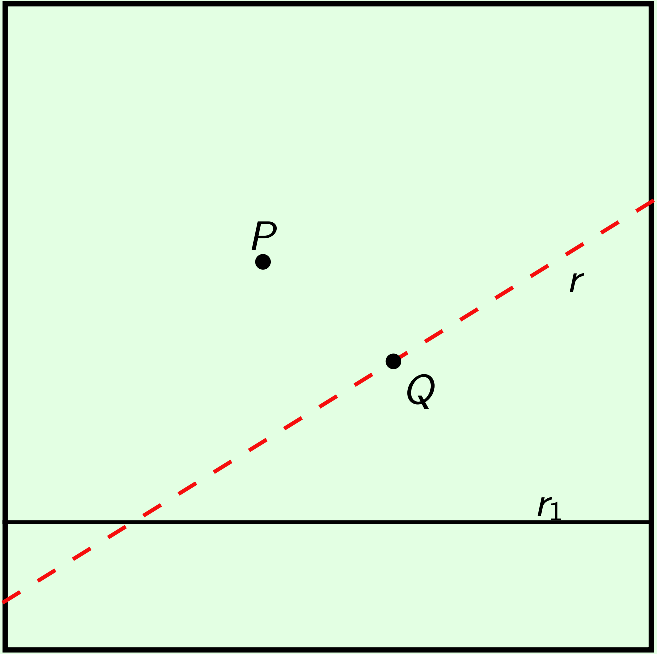
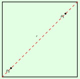
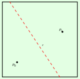
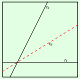
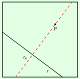
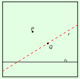
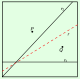


P
r
Q
r1

| | |
|---|---|
| Line between two points<br> | Axe of a segment<br> |
| Bisector of an angle<br> | Perpendicular through a point<br> |
| Tg of a parabola<br> | Common tg of two parabola<br> |
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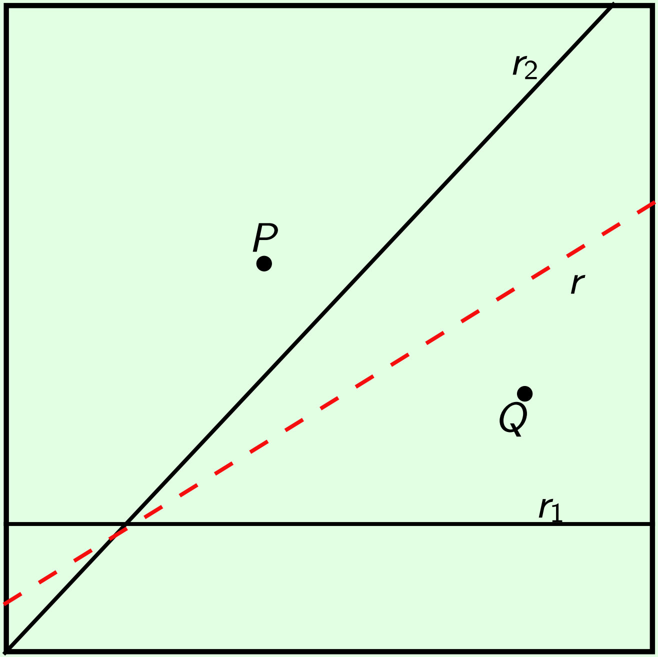
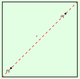
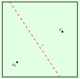
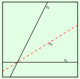
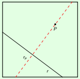
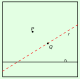
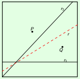
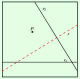
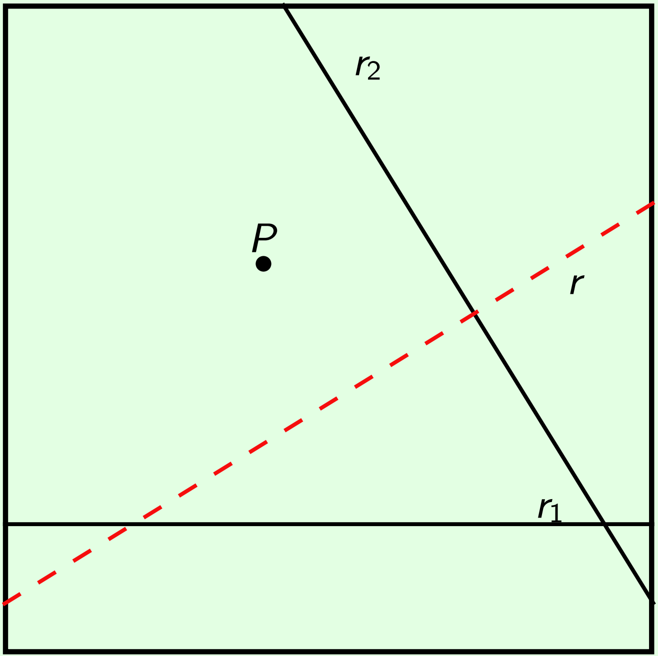
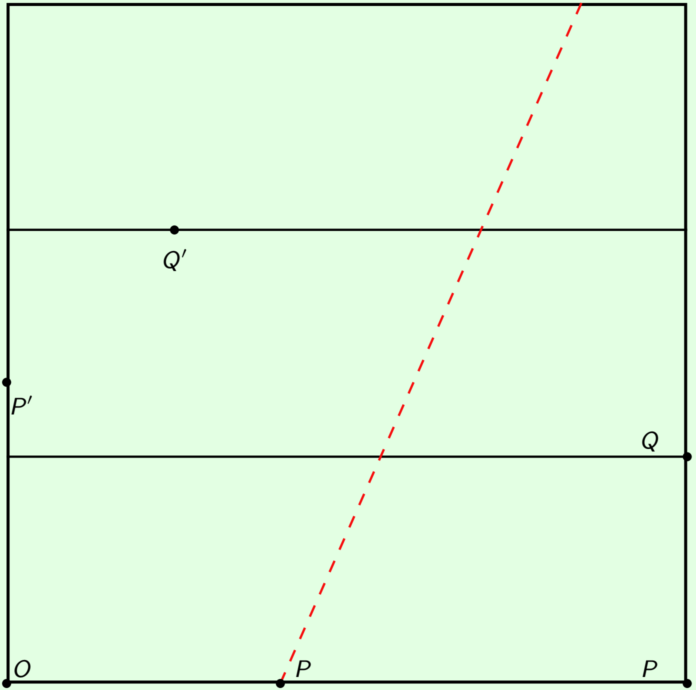


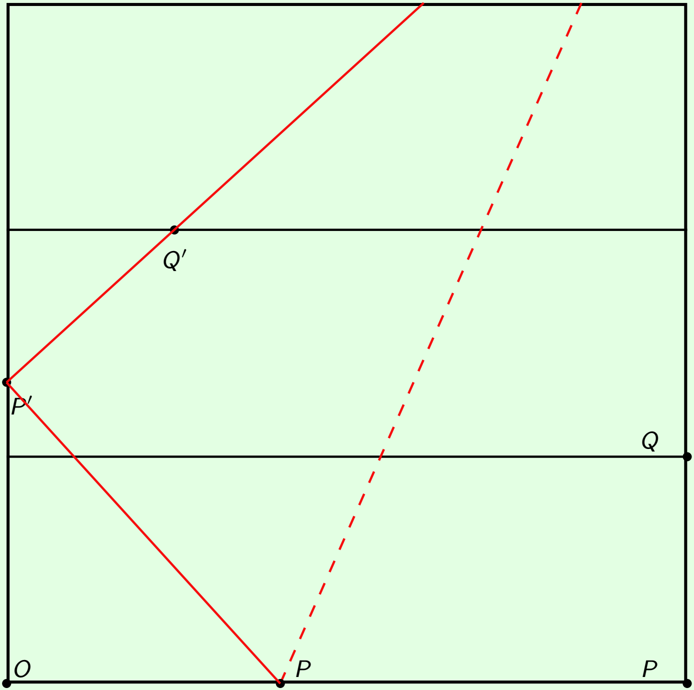
$r_2$
$P$
$r$
$Q$
$r_1$

| | |
|---|---|
| Line between two points<br> | Axe of a segment<br> |
| Bisector of an angle<br> | Perpendicular through a point<br> |
| Tg of a parabola<br> | Common tg of two parabola<br> |
| Tangent of a parabola and perpendicular to a line<br> | |


$r_2$
$P$
$r$
$r_1$



Q′
P′
Q
O
P
P



Q′
P′
Q
O
P
P

- $\sqrt[3]{2}$ is NOT an origami number in the sense of Auckley-Cleveland.
- $\sqrt[3]{2}$ is an origami numberi in the sense of Hatori-Huzita.

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References:


- D.Auckly, J.Cleveland, *Totally real origami and impossible paper folding*, American Mathematics Monthly 102 (1995)
- S. Yin, *The mathematics of origami*, 2009
- Centro diffusione origami:http://www.origami-cdo.it
- An artist Robert J. Lang: www.langorigami.com

- Me amorvincomni@gmail.com: I have a dropbox folder and under request I will share it.

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References:


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